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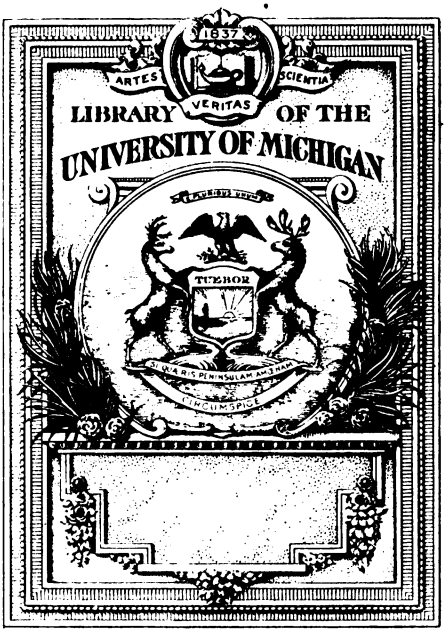
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TRIGONOMETRY,

PLANE

AND

SPHERICAL;

WITH THE

CONSTRUCTION and APPLICATION

OF

LOGARITHMS.

By *THOMAS SIMPSON*, 1710-1761 F.R.S.

The FIFTH EDITION.

LONDON:

Printed for F. WINGRAVE, Succeſſor to
Mr. NOURSE, in the Strand;

By LUKE HANSARD, N° 6, Great Turnſtile, Lincoln's Inn Fields.

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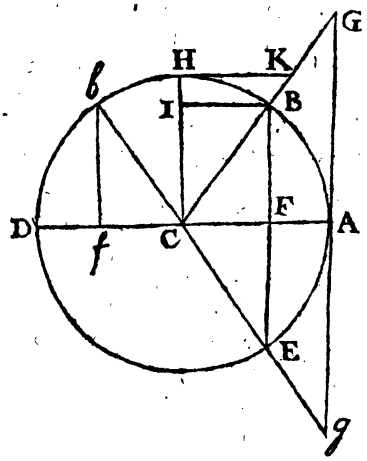
Plane Trigonometry.

DEFINITIONS.

1. PLANE Trigonometry is the art whereby, having given any three parts of a plane triangle, (except the three angles) the reſt are determined. In order to which, it is not only requiſite that the peripheries of circles, but alſo certain right-lines in, and about, the circle, be ſuppoſed divided into ſome aſſigned number of equal parts.

2. The periphery of every circle is ſuppoſed to be divided into 360 equal parts, called degrees; and each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called ſeconds, or ſecond minutes, &c.



3. Any part AB of the periphery of the circle is called an arch, and is ſaid to be the meaſure of the angle ACB at the center, which it ſubtends.

Note, *The degrees, minutes, ſeconds, &c. contained in any arch, or angle, are wrote in this manner,* 50° 18′ 35″, *which ſignifies that the given arch, or angle, contains* 50 *degrees,* 18 *minutes, and* 35 *ſeconds.*

4. The difference of any arch from 90° (or a quadrant) is called its complement; and its difference from 180° (or a ſemicircle) its ſupplement.

5. A chord, or ſubtenſe, is a right-line drawn from one extremity of an arch to the other: thus the right line BE is the chord, or ſubtenſe, of the arch BAE or BDE.

6. The ſine, or right-ſine, of an arch, is a right line drawn from one extremity of the arch, perpendicular to the diameter paſſing through the other extremity. Thus BF is the ſine of the arch AB or DB.

7. The verſed ſine of an arch is the part of the diameter intercepted between the ſine and the periphery. Thus AF is the verſed ſine of AB; and DF of DB.

8. The

8. The co-ſine of an arch is the part of the diameter intercepted between the center and ſine; and is equal to the ſine of the complement of that arch. Thus CF is the co-ſine of the arch AB, and is equal to BI, the ſine of its complement HB.

9. The tangent of an arch is a right line touching the circle in one extremity of that arch, produced from thence till it meets a right-line paſſing through the center and the other extremity. Thus AG is the tangent of the arch AB.

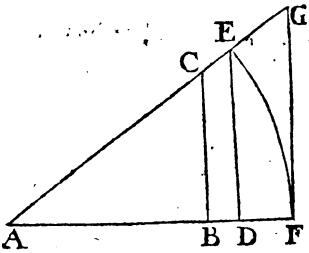
10. The ſecant of an arch is a right-line reaching, without the circle, from the center to the extremity of the tangent. Thus CG is the ſecant of AB.

11. The co-tangent, and co-ſecant, of an arch are the tangent, and ſecant, of the complement of that arch. Thus HK and CK are the co-tangent and co-ſecant of AB,

12. A trigonometrical canon is a table exhibiting the length of the ſine, tangent, and ſecant, to every degree and minute of the quadrant, with reſpect to the radius; which is ſuppoſed unity, and conceived to be divided into 10000, or more, decimal parts. By the help of this table, and the doctrine of ſimilar triangles, the whole buſineſs of trigonometry is performed; which I ſhall now proceed to ſhew. But, firſt of all, it will be proper to obſerve, that the ſine of any arch A*b* greater than 90°, is equal to the ſine of another arch AB as much below 90°; and that, its co-ſine C*f*, tangent A*g*, and ſecant C*g*, are alſo reſpectively equal to the co-ſine, tangent, and ſecant of its ſupplement AB; but only are negative, or fall on contrary ſides of the points C and A, from whence they have their origin. All which is manifeſt from the definitions.

THEOREM I.

In any right-angled plane triangle ABC, *it will be as the hypothenuſe is to the perpendicular, ſo is the radius (of the table) to the ſine of the angle at the baſe.*



For, let AE or AF be the radius to which the table of ſines, &c. is adapted, and ED the ſine of the angle A or arch EF *(Vid. Def. 3. and 6.)*; then, becauſe of the ſimilar triangles ACB and AED, it will be AC : BC : : AE : ED *(by* 14. 4.) *Q. E. D.*

Thus, if AC = ,75, and BC = ,45; then it will be, ,75 : ,45 : : 1 (radius) : the ſine of A = 6; which, in the table, anſwers to 36° 52′, the meaſure, or value of A.

THEOREM II.

In any right-angled plane triangle ABC, *it will be, as the baſe* AB *is to the perpendicular* BC, *ſo is the radius (of the table) to the tangent of the angle at the baſe.*

For, let AE or AF be the radius of the table, or canon *(ſee the preceding figure)*, and FG the tangent of the angle A, or arch EF *(Vid. Def. 3. and 9.)*; then, by reaſon of the ſimilarity of the triangles ABC, AFG, it will be, AB : BC : : AF : FG. *Q. E. D.*

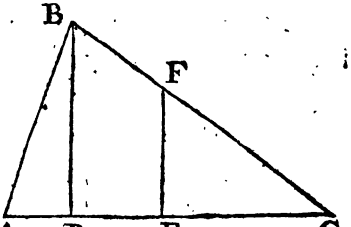
Note, *In the quotations where you meet with two numbers, (as* 14. 4 *) without any mention of Prop Theor. &c. reference is made to the* Elements of Geometry *publiſhed by the ſame author; to which this little tract is deſigned as an* Appendix.

Thus

Thus let AB = ,8, and BC = ,5; then we shall have ,8 : ,5 : : 1 (radius) : tangent A = ,625; whence A itself is found, by the canon, to be 32° 00′.

THEOREM III.

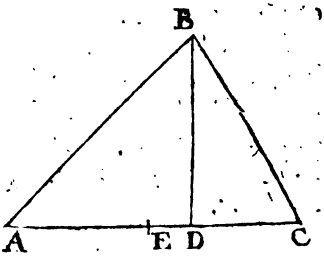
In every plane triangle ABC, *it will be, as any one side is to the sine of its opposite angle, so is any other side to the sine of its opposite angle.*



For take CF = AB, and upon AC let fall the perpendiculars BD and FE; which will be the sines of the angles A and C to the equal radii AB and CF. Now the triangles CBD, CFE being similar, we have CB : BD (*sin.* A) : : CF (AB) : FE (*sin.* C). *Q. E. D.*

THEOREM IV.

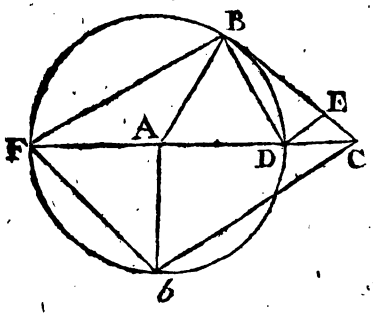
As the base of any plane triangle ABC, *is to the sum of the two sides, so is the difference of the sides to twice the distance* DE *of the perpendicular from the middle of the base.*



For (*by Cor. to* 9. 2.) $\overline{AB+BC} \times \overline{AB-BC} = AC \times 2DE$; whence AC : AB + BC : : AB — BC : 2DE (*by* 10. 4.) *Q. E. D.*

THEOREM V.

In any plane triangle, it will be, as the ſum of any two ſides is to their difference, ſo is the tangent of half the ſum of the two oppoſite angles, to the tangent of half their difference.



For, let ABC be the triangle, and AB and AC the two propoſed ſides; and from the center A, with the radius AB, let a circle be deſcribed, interſecting CA produced, in D and F; ſo that CF may expreſs the ſum, and CD the difference, of the ſides AC and AB: join F, B and B, D, and draw DE parallel to FB, meeting BC in E.

Then, becauſe 2ADB = ADB + ABD (*by* 12. 1.) = C + ABC (*by* 9. 1.) it is plain that ADB is equal to half the ſum of the angles oppoſite to the ſides propoſed. Moreover, ſince ABC = ABD (ADB) + DBC, and C = ADB — DBC (*by* 9. 1.) it is plain that ABC — C is = 2DBC; or that DBC is equal to half the difference of the ſame angles.

Now, becauſe of the parallel lines BF and ED, it will be CF : CD : : BF : DE; but BF and DE, becauſe DBF and BDE are right-angles (*by* 13. 3. *and* 7. 1.) will be tangents of the foreſaid angles FDB (ADB) and DBE (DBC) to the radius BD. *Q. E. D.*

COROL-

COROLLARY.

Hence, in two triangles ABC and A*b*C, having two ſides equal, each to each, it will be (*by equality*), as tang. $\frac{A\mathit{b}C+AC\mathit{b}}{2}$: tang. $\frac{A\mathit{b}C-AC\mathit{b}}{2}$ :: tang. $\frac{ABC+ACB}{2}$: tang. $\frac{ABC-ACB}{2}$. But, if CA*b* be ſuppoſed a right-angle, then will A*b*C + AC*b* alſo = a right angle (*by Cor.* 3. *to* 10. 1.) and the tangent of $\frac{A\mathit{b}C+AC\mathit{b}}{2}$ = radius. Therefore in this caſe our proportion will become,

Radius : tang. $\frac{A\mathit{b}C-AC\mathit{b}}{2}$ ($=A\mathit{b}C-45°$) :: tang. $\frac{ABC+ACB}{2}$: tang. $\frac{ABC-ACB}{2}$. Which gives the following *Theorem*, for finding the angles oppoſite to any two propoſed ſides; the included angle, and the ſides themſelves, being known.

As the leſſer of the propoſed ſides (A*b* *or* AB) *is to the greater* (AC), *ſo is radius to the tangent of an angle* (A*b*C, ſee Theor. 2.) *And as radius to the tangent of the exceſs of this angle above* 45°, *ſo is the tangent of half the ſum of the required angles to the tangent of half their difference**.

* *This Theorem, though it requires two proportions, is commonly uſed by Aſtronomers in determining the elongation and parallaxes of the planets* (*being beſt adapted to logarithms*); *for which reaſon it is here given.*

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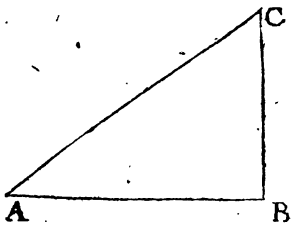
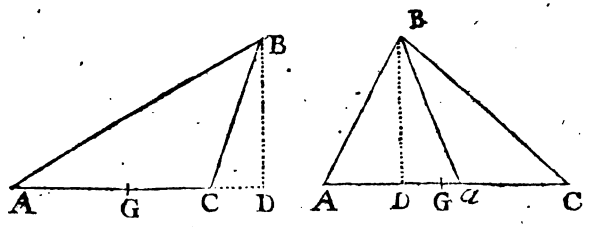


The ſolution of the caſes of right-angled plane triangles.

Caſe.	Given	Sought	Solution.
1	The hyp. AC and the angles	One leg BC	As radius is to the ſine of A, ſo is the hyp. AC to the leg BC (*by Theor. I.*)
2	The hyp. AC and one legBC	The angles	As AC : BC : : radius : ſin. A (*Theor. I.*) whoſe complement is the angle C.
3	The hyp. AC. and one legBC	The other leg AB	Let the angles be found, *by Caſe* 2. and then the required leg AB. *by Caſe* 1.
4	The angles and one legBC	The hyp. AC	As ſine A : radius : : the leg BC : to the hyp. AC (*Theor. I.*)
5	The angles and one leg BC	The other leg AB	As ſine A : BC : : ſine C : AB (*by Theor. III.*) Or, as radius : tang. C : : BC : AB (*by Theor. II.*)
6	The two legs AB and BC	The angles	As AB : BC : : radius : tang. A (*by Theorem II.*) whoſe complement is the angle C.
7	The two legs AB and BC	The hyp. AC	Let the angles be found, *by Caſe* 6. and then the hyp. AC, *by Caſe* 4.



The solution of the cases of oblique plane triangles.

Case.	Given	Sought	Solution.
1	The angles and one side AB	Either of the other sides BC	As sine C : AB :: sine A : BC (*by Theor. III.*)
2	Two sides AB, BC and an ang. C op. to one of 'em.	The other angles A and ABC	As AB : sin. C :: BC : sin. A (*by Theor. III.*) which added to C, and the sum subtracted from 180 gives the other angle ABC.
3	Two sides AB, BC and an opp. angle C	The other side AC	Let the angle ABC be found, by the preceding case, and then it will be, sin. C : AB :: sin. ABC : AC (*by Theor. III.*)
4	Two sides AC, AB and the included angle A	The other angles C and ABC	As sum of AB and AC : their dif. :: tang. of half the sum of ABC and C : tang. of half their diff. (*by Theor. V.*) which added to, and subtracted from, the half sum, gives the two angles.
5	Two sides AC, AB and the incl ∠ A.	The other side BC	Let the angles be found by the last case, and then BC, *by case* 1.
6	All the three sides.	An angle, suppose A	Let fall a perp. BD opp. to the req. angle: then (*by Theor. IV.*) as AC : sum of AB and BC :: their dif. : dist. DG of the perp. from the middle of the base; whence, AD being also known, the angle A will be found by *Case* 2. of right-angles.

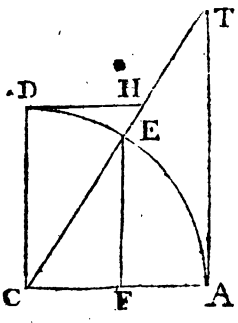
Note,

Note, The 2d and 3d caſes are ambiguous, or admit of two different anſwers each, when the ſide AB oppoſite the given angle C *(ſee fig. 2.)* is leſs than the given ſide BC, adjacent to it (except the angle found is exactly a right one): for then another right-line B*a*, equal to BA, may be drawn from B to a point in the baſe, ſomewhere between C and the perpendicular BD, and therefore the angle found by the proportion AB (*a*B) : ſin. C. : : BC : ſin. A *(or of* C*a*B,) may, it is evident, be either the acute angle A, or the obtuſe one C*a*B (which is its ſupplement), the ſines of both being exactly the ſame.

Having laid down the method of reſolving the different caſes of plane triangles, by a table of ſines and tangents; I ſhall here ſhew the manner of conſtructing ſuch a table (as the foundation upon which the whole doctrine is grounded); in order to which, it will be requiſite to premiſe the following propoſitions.

Proposition I.

The ſine of an arch being given, to find its co-ſine, verſed ſine, tangent, co-tangent, ſecant, and co-ſecant.



Let AE be the propoſed arch, EF its ſine, CF its co-ſine, AF its verſed ſine, AT its tangent, CT its ſecant, DH its co-tangent, and CH its co-ſecant.

Then (*by* 8. 2.) we have CF = the ſquare root of $CE^2 - EF^2$; whence, not only the co-ſine CF, but alſo the verſed ſine AF, will

will be known. Then becauſe of the ſimilar triangles CFE, CAT, and CDH, it will be (*by* 14. 4.)

1. CF : FE : : CA : CT; whence the tangent is known.

2. CF : CE (CA) : : CA : CT; whence the ſecant is known.

3. EF : CF : : CD : DH; whence the co-tangent is known.

4. EF : EC (CD) : : CD : CH; whence the co-ſecant is known.

Hence it appears,

1. That the tangent is a fourth proportional to the co-ſine, the ſine, and radius.

2. That the ſecant is a third-proportional to the co-ſine and radius.

3. That the co-tangent is a fourth proportional to the ſine, co-ſine, and radius.

4. And that the co-ſecant is a third proportional to the ſine and radius.

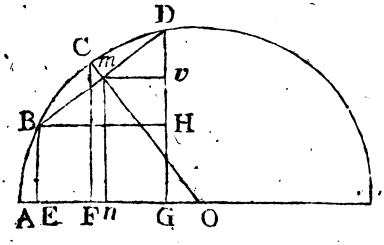
5. It appears moreover (becauſe AT : AC : : CD (AC) : DH), that the rectangle of the tangent and co-tangent is equal to the ſquare of the radius (*by* 10. 4.): whence it likewiſe follows, that the tangent of half a right angle is equal to the radius; and that the co-tangents of any two different arches (repreſented by P and Q) are to one another, inverſely as the tangents of the ſame arches: for, ſince tang. P × co-tang. P = ſqu. rad. = tang. Q × co-tang. Q; therefore will co-

tang.

tang. P : co-tang. Q :: tang. Q : tang. P; or as co-tang. P : tang. Q :: co-tang. Q : tang. P (*by* 10. 4.)

Prop. II.

If there be three equidifferent arches AB, AC, AD, *it will be, as radius is to the co-ſine of their common difference* BC, *or* CD, *ſo is the ſine* CF, *of the mean, to half the ſum of the ſines* BE+DG, *of the two extremes: and, as radius to the ſine of the common difference, ſo is the co-ſine* FO *of the mean, to half the difference of the ſines of the two extremes.*



For let BD be drawn, interſecting the radius OC in *m*; alſo draw *mn* parallel to CF, meeting AO in *n*; and BH and *mv*, parallel to AO, meeting DG in H and *v*.

Then, the arches BC and CD being equal to each other (*by hypotheſis*), OC is not only perpendicular to the chord BD, but alſo biſects it (*by* 1. 3.) and therefore B*m* (*or* D*m*) will be the ſine of BC (*or* DC), and O*m* its co-ſine: moreover *mn*, being an arithmetical mean between the ſines BE, DG of the two extremes (becauſe B*m* = D*m*) is therefore equal to half their ſum, and D*v* equal to half their difference. But, becauſe of the ſimilar triangles OCF, O*mn* and D*vm*,

It will be $\left\{\begin{matrix} \text{OC} : \text{O}m :: \text{CF} : mn \\ \text{OC} : \text{D}m :: \text{FO} : \text{D}v \end{matrix}\right\}$ *Q. E. D.*

Corol-

Corollary I.

Becauſe of the foregoing proportions, we have $mn\left(\frac{DG+BE}{2}\right)=\frac{Om\times CF}{OC}$*, and $Dv\left(\frac{DG-BE}{2}\right)=\frac{Dm\times FO}{OC}$; and therefore $DG+BE=\frac{2Om\times CF}{OC.}$ and $DG-BE=\frac{2Dm\times FO.}{OC.}$

Corollary II.

Hence, if the mean arch AC be ſuppoſed that of 60°; then OF being the co-ſine of 60°, = ſine 30° = ½ chord of 60° = ½ OC, it is manifeſt that DG—BE will, in this caſe, be barely = D*m*; and conſequently DG = D*m* + BE. From whence, and the preceding corollary, we have theſe two uſeful theorems.

1. *If the ſine of the mean, of three equidifferent arches (ſuppoſing radius unity) be multiplied by twice the co-ſine of the common difference, and the ſine of either extreme be ſubtracted from the product, the remainder will be the ſine of the other extreme.*

2. *The ſine of any arch, above* 60 *degrees, is equal to the ſine of another arch, as much below* 60°, *together with the ſine of its exceſs above* 60°.

* Note, $\frac{Om\times CF}{OC}$, *taken in a* geometrical ſenſe, *denotes a fourth-proportional to* OC, Om *and* CF; *but,* arithmetically, *it ſignifies the quantity ariſing by dividing the product of the meaſures of* Om *and* CF *by that of* OC. *Underſtand the like of others.*

Prop.

PROP. III.

To find the ſine of a very ſmall arch; ſuppoſe that of 15′.

It is found, in p. 181. of the Elements, that the length of the chord of $\frac{1}{384}$ of the ſemi-periphery is expreſſed by ,00818121 (radius being unity); therefore, as the chords of very ſmall arches are to each other nearly as the arches themſelves (*vid. p.* 181.) we ſhall have, as $\frac{1}{384} : \frac{1}{360} :: ,00818121 : ,008726624$, the chord of $\frac{1}{360}$, or half a degree; whoſe half, or ,00436312, is therefore the ſine of 15′, very nearly.

From whence the ſine of any inferior arch may be found by bare proportion. Thus, if the ſine of 1′ be required, it will be, 15′ : 1′ : : ,00436312 : ,000290888, the ſine of the arch of one minute, nearly.

But if you would have the ſine of 1′ more exactly determined (from which the ſines of other arches may be derived with the ſame degree of exactneſs); then let the operations, in p. 181, be continued to 11 biſections, and a greater number of decimals be taken; by which means you will get the chord of $\frac{1}{6144}$ part of the ſemi-periphery to what accuracy you pleaſe: then, by proceeding as above (for finding the ſine of 1′), the ſine of 1 minute will alſo be obtained to a very great degree of exactneſs.

PROP. IV.

To ſhew the manner of conſtructing the trigonometrical canon.

Firſt, find the ſine of an arch of one minute, by the preceding Prop. and then its co-ſine, by

Prop.

Prop. 1. which let be denoted by C; then (*by Theor.* 1. *p.* 13.) we shall have

$2C \times \text{fine } 1' - \text{fine } 0' = \text{fine } 2'.$
$2C \times \text{fine } 2' - \text{fine } 1' = \text{fine } 3'.$
$2C \times \text{fine } 3' - \text{fine } 2' = \text{fine } 4'.$
$2C \times \text{fine } 4' - \text{fine } 3' = \text{fine } 5'.$

And thus are the sines of 6′, 7′, 8′, &c. successively derived from each other.

The sines of every degree and minute, up to 60°, being thus found; those of above 60° will be had by addition only (*from Theor.* 2. *p.* 15.) then, the sines being all known, the tangents and secants will likewise become known, *by Prop.* 1.

Note, If the sine of every 5th minute, *only*, be computed according to the foregoing method, the sines of all the intermediate arches may be had from thence, by *barely* taking the proportional parts of the differences, and that so near as to give the first six places true in each number; which is sufficiently exact for all common purposes.

Scholium.

Although what has been hitherto laid down for constructing the trigonometrical-canon, is abundantly sufficient for that purpose, and is also very easily demonstrated; yet, as the first sine, from whence the rest are all derived, must be carried on to a great number of places, to render the numerous deductions from it but tolerably exact (because in every operation the error is multiplied), I shall here subjoin a different method, which will be found to have the advantage, not only in that, but in many other respects.

First, then, from the co-sine of 15°, which is given (*by p.* 181 *of the Elements*) $= \frac{1}{4}\sqrt{2+\sqrt{3}}$ $=,965925826$, &c. ($= \frac{1}{2}$ the supplement chord of 30°) and the sine of 18°, which is $= \frac{1}{4}\sqrt{5}-\frac{1}{4} =$

,309017, &c. (equal to half the side of a decagon inscribed in the circle) let the co-sine of 3°, the difference between 18° and 15°, be found *; from which the co-sine of 45′ will be had, by two bisections only: whence the sines of all the arches in the progression 1° 30′, 2° 15′, 3° 00′, 3° 45′, &c. may be determined (*by Theor.* 1. *p.* 15.) and that to any assigned degree of exactness.

The sines of all the terms of the progression 45′, 1° 30′, 2° 15′, &c. up to 60°, being thus derived, the next thing is to find, by help of these, the sines of all the intermediate arches, to every single minute.

This, if you desire no more than the 4 or 5 first places of each (which is exact enough where nothing less than degrees and minutes is regarded), may be effected by barely taking the proportional parts of the differences.

But if a greater degree of accuracy be insisted on, and you would have a table carried on to 7 or 8 places, each number (which is sufficient to give the value of an angle to seconds, and even to thirds, in most cases) then the operation may be as follows:

1°. Multiply the sum of the sines of any two adjacent terms of the progression 45′, 1° 30′, 2° 15′, 3° 00′, 3° 45′, &c. (betwixt which you would find all the intermediate sines) by the fraction ,000000423, for a first product; and this, again, by 22, for a second product; to which last, let $\frac{1}{45}$ of the difference of the two proposed sines (or extremes) be added, and the sum will be the excess of the first of the intermediate sines above the lesser extreme.

* Note, *The co-sine of the difference of two arches* (supposing radius unity), *is found by adding the product of their sines to that of their co-sines; as is hereafter demonstrated.*

2°. From this excefs let the firſt product be continually ſubtracted; that is, firſt, from the excefs itſelf; then from the remainder; then from the laſt remainder, and ſo on 44 times.

3°. To the leſſer extreme add the forementioned excefs; and, to the ſum, add the firſt remainder; to this ſum add the next remainder, and ſo on continually: then the ſeveral ſums thus ariſing will reſpectively exhibit the ſines of all the intermediate arches, to every ſingle minute, excluſive of the laſt; which, if the work be right, will agree with the greater extreme itſelf, and therefore will be of uſe in proving the operation.

But to illuſtrate the matter more clearly, let it be propoſed to find the ſines of all the intermediate arches between 3° 00′ and 3° 45′ to every ſingle minute, thoſe of the extremes being given, from the foregoing method, equal to ,05233595 and ,06540312 reſpectively. Here, the ſum of the ſines of the extremes being multiplied by ,000000423, the firſt product will be ,00000000498, &c. or ,000000050, *nearly* (which is ſufficiently exact for the preſent purpoſe); and this, again, multiplied by 22, gives ,00000011 for a 2d product; which added to ,000290815, $\frac{1}{45}$ part of the difference of the two given extremes, will be ,000290915, the excefs of the ſine of 3° 01′ above that of 3° 00′. From whence, by proceeding according to the 2d and 3d rules, the ſines of all the other intermediate arches are had, by addition and ſubtraction only. See the operation.

,000290	4915	exceſs	,05233595	ſine 3° 0′
,000000	0050		,0002904915	
	4865	1st rem.	,0526264415	ſine 3° 1′
	50		2904865	
	4815	2d rem.	,0529169280	ſine 3° 2′
	50		2904815	
	4765	3d rem.	,0532074095	ſine 3° 3′
	50		2904765	
	4715	4th rem.	,0534978860	ſine 3° 4′
	50		2904715	
	4665	5th rem.	,0537883575	ſine 3° 5′
	50		2904665	
	4615	6th rem.	,0540788240	ſine 3° 6′
	50		2904615	
	4565	7th rem.	,0543692855	ſine 3° 7′
	50		2904565	
	4515	8th rem.	,0546597420	ſine 3° 8′
	50		2904515	
	4465	9th rem.	,0549501935	ſine 3° 9′
	50		2904465	
	4415	10th rem.	,0552406400	ſine 3° 10′
	&c.		&c.	

Again, as a ſecond example, let it be required to find the ſines of all the arches, to every minute, between 59° 15′ and 60° 00′; thoſe of the two extremes being firſt found, by the preceding method.

method. In this case, the two extremes, being ,85940641 and ,86602540, their sum will be = 1,72543, &c. and their difference = ,00661899; whereof the former, multiplied by ,0000000423 *(see the rule)* gives ,00000007298, &c. or ,00000007 30, nearly, for the first product (which is exact enough for our purpose); therefore the 2d product, or ,00000007 30 × 22, will be ,0000016060; which, added to $\frac{1}{44}$ of the difference, gives ,0001486947; from whence the operation will be as follows:

,00014	86947	exceſs	,85940641	ſine 59° 15′
,00000	00730	1st prod.	0001486947	
	86217	1st rem.	,8595551047	ſine 59° 16′
	730		1486217	
	85487	2d rem.	,8597037264	ſine 59° 17′
	730		1485487	
	84757	3d rem.	,8598522751	ſine 59° 18′
	730		1484757	
	84027	4th rem.	,8600007508	ſine 59° 19′
	730		1484027	
	83297	5th rem.	,8601491535	ſine 59° 20′
	730		1483297	
	82567	6th rem.	,8602974832	ſine 59° 21′
			1482567	
			,8604457399	ſine 59° 22′
	&c.		&c.	

After the same manner the sines of all the intermediate arches between any other two proposed extremes may be derived, even up to 90 degrees;

but thoſe of above 60° are beſt found from thoſe below, as has been ſhewn elſewhere.

The reaſons upon which the foregoing operations are founded, depend upon principles too foreign from the main deſign of this treatiſe, to be explained here, (even would room permit); however, as to the correctneſs and utility of the method itſelf, I will venture to affirm, that, whoever has the inclination, either to calculate new tables, or to examine thoſe already extant, will not find one quarter of the trouble, this way, as he unavoidably muſt according to the common methods.

Spherical Trigonometry.

DEFINITIONS.

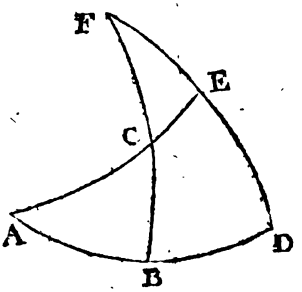
1. A Great-circle of a ſphere is a ſection of the ſphere by a plane paſſing thro' the center.

2. The axis of a great-circle is a right-line paſſing through the center, perpendicular to the plane of the circle: and the two points, where the axis interſects the ſurface of the ſphere, are called the poles of the circle.

3. A ſpherical angle is the inclination of two great-circles.

4. A ſpherical triangle is a part of the ſurface of the ſphere included by the arches of three great-circles; which arches are called the ſides of the triangle.

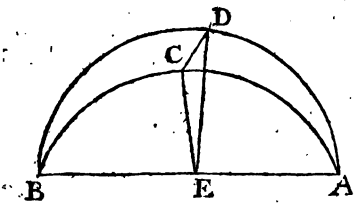
5. If thro' the poles A and F of two great-circles DF and DA, ſtanding at right-angles, two other great-circles ACE and FCB be conceived to paſs, and thereby form two ſpherical triangles ABC and FCE, the latter of the triangles ſo formed is ſaid to be the complement of the former; and *vice verſa*.



COROLLARIES.

1. It is manifeſt *(from Def.* 1.*)* that the ſection of two great-circles (as it paſſes through the center) will be a diameter of the ſphere; and conſequently, that their peripheries will always interſect each other in two points at the diſtance of a ſemicircle, or 180 degrees.

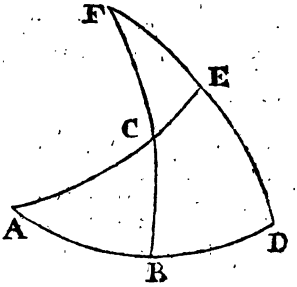
2. It alſo appears *(from Def.* 2.*)* that all great-circles, paſſing through the pole of a given circle, cut that circle at right-angles; becauſe they paſs through, or coincide with the axis, which is perpendicular to it.



3. It follows moreover, that the periphery of a great-circle is every where 90 degrees diſtant from its pole; and that the meaſure of a ſpherical angle CAD* is an arch of a great-circle intercepted by the two circles ACB, ADB forming that angle, and whoſe pole is the angular point A. For let the diameter AB be the interſection of the great-circles ADB and ACB *(ſee Corol.* 1.*)* and let the plane, or great-circle, DEC be conceived perpendicular to that diameter, interſecting the ſurface of the ſphere in the arch CD; then it is manifeſt that AD=BD=90°, and AC=BC=90°. *(Corol.* 1.*)* and that CD is the meaſure of the angle DEC *(or* CAD*)* the inclination of the two propoſed circles.

* Note, *Although a ſpherical angle is, properly, the inclination of two great-circles, yet it is commonly expreſſed by the inclination of their peripheries at the point where they interſect each other.*

4. Hence

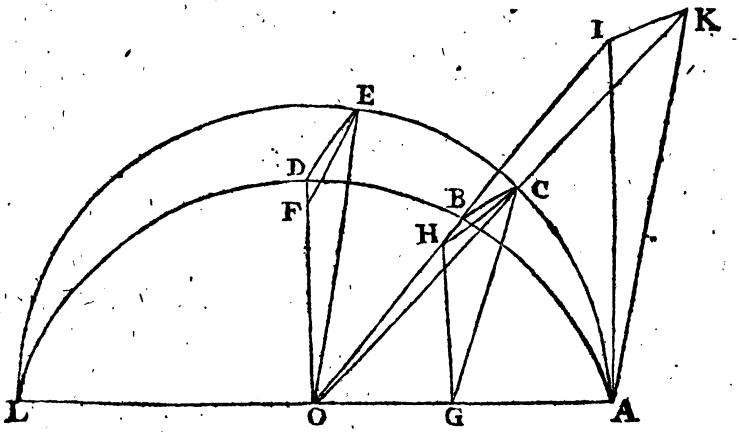


4. Hence it is alſo manifeſt, that the angles B and E, of the complemental triangles ABC and FCE, are both right angles; and that CE is the complement of AC, CF of BC, BD *(or the angle* F) of AB and EF of ED *(or the angle* A).

THEOREM I.

In any right-angled ſpherical triangle it will be, as radius is to the ſine of the angle at the baſe, ſo is the ſine of the hypothenuſe to the ſine of the perpendicular; and as radius to the co-ſine of the angle at the baſe, ſo is the tangent of the hypothenuſe to the tangent of the baſe.

DEMONSTRATION.



Let ADL and AEL be two great-circles of the ſphere interſecting each other in the diameter AL, making

making an angle DOE, measured by the arch ED; the plane DOE being supposed perpendicular to the diameter AL, at the center O.

Let AB be the base of the proposed triangle, BC the perpendicular, AC the hypothenuse, and BAC (*or* DAE=DE=DOE) the angle at the base: moreover, let CG be the sine of the hypothenuse, AK its tangent, AI the tangent of the base, CH the sine of the perpendicular, and EF the sine of the angle at the base; and let I, K and G, H be joined.

Because CH is perpendicular to the plane of the base (*or paper*), it is evident, that the plane GHC will be perpendicular to the plane of the base, and likewise perpendicular to the diameter AL, because GC, being the sine of AC, is perpendicular to AL. Moreover, since both the planes OIK and AIK are perpendicular to the plane of the base (*or paper*), their intersection IK will also be perpendicular to it, and consequently the angle AIK a right-angle. Therefore, seeing the angles OFE, GHC and AIK are all right angles, and that the planes of the three triangles OFE, GHC and AIK are all perpendicular to the diameter AL, we shall, *by similar triangles*,

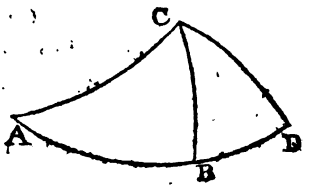
have $\left\{\begin{array}{l} OE : EF :: GC : CH \\ OE : OF :: AK : AI \end{array}\right\}$

that is, $\left\{\begin{array}{l} \text{Radius : sine of EOF (}or\text{ BAC) :: sine of AC : sine of BC.} \\ \text{Radius : co-sine of EOF (}or\text{ BAC) :: tang. AC : tang. AB.} \end{array}\right.$ *Q. E. D.*

Corollary 1.

Hence it follows, that the sines of the angles of any oblique spherical triangles ADC are to one another, directly, as the sines of the opposite sides.

For



For let BC be perpendicular to AD;
then since { radius : sine A :: sine AC : sine BC } *by the former part of the theorem*; we shall have, sine A × sine AC (= radius × sine BC) = sine D × sine DC (*by* 10. 4.) and consequently sine A : sine D :: sine DC : sine AC; or sine A : sine DC :: sine D : sine AC.

Corollary 2.

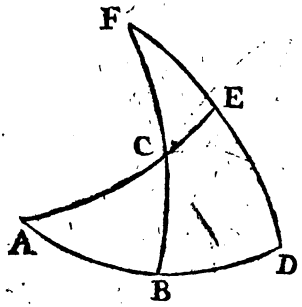
It follows, moreover, that, in right-angled spherical triangles ABC, DBC, having one leg BC common, the tangents of the hypothenuses are to each other, inversely, as the co-sines of the adjacent angles.
For since { radius : co-sine ACB :: tan. AC : tan. BC } *by the latter part of the theorem*; we shall (*by arguing as above*) have co-sine ACB : co-sine DCB :: tang. DC : tang. AC.

Theorem II.

In any right-angled spherical triangle (ABC) *it will be, as radius is to the co-sine of one leg, so is the co-sine of the other leg to the co-sine of the hypothenuse.*

Demon-

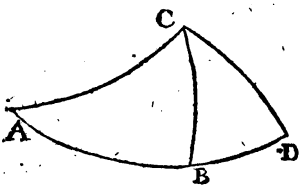
DEMONSTRATION.



Let CEF be the complemental-triangle to ABC, according to what has been already ſpecified; then it will be, *by Theor.* 1. *Caſe* 1.

Radius : ſine F : : ſine CF : ſine CE; that is,
Radius : co-ſine BA : : co-ſine CB : co-ſine AC
(See Cor. 4. *p.* 25.) *Q. E. D.*

COROLLARY.



Hence, if two right-angled ſpherical triangles ABC, CBD have the ſame perpendicular BC, the co-ſines of their hypothenuſes will be to each other, directly, as the co-ſines of their baſes.

For { rad : co-ſin. BC : : co-ſin. AB : co-ſine AC,
ſince { rad : co-ſin. BC : : co-ſin. DB : co-ſine DC,
therefore, by equality and permutation, co-ſine AB : co-ſine DB : : co-ſine AC : co-ſine DC.

THEOREM III.

In any right-angled ſpherical triangle (ABC) *it will be, as radius is to the ſine of either angle, ſo is the co-ſine of the adjacent leg to the co-ſine of the oppoſite angle.*

DEMON-

DEMONSTRATION.

Let CEF be as in the preceding propoſition; then, *by Theor.* 1. *Caſe* 1. it will be, radius : ſine C : : ſine CF : ſine EF; that is, radius : ſine C : : co-ſine BC : co-ſine A. *Q. E. D.*

COROLLARY.

Hence, in right-angled ſpherical triangles ABC, CBD, having the ſame perpendicular BC *(ſee the laſt figure)*, the co-ſines of the angles at the baſe will be to each other, directly, as the ſines of the vertical angles:

For { radius : ſine BCA : : co-ſine CB : co-ſine A,
ſince { radius : ſine BCD : : co-ſine CB : co-ſine D,
therefore, *by equality and permutation,*

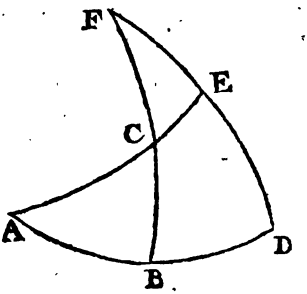
Co-ſine A : co-ſine D : : ſine BCA : ſine BCD.

THEOREM IV.

In any right-angled ſpherical triangle (ABC) *it will be, as radius is to the ſine of the baſe, ſo is the tangent of the angle at the baſe to the tangent of the perpendicular.*

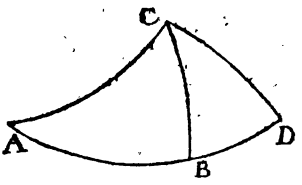
For, ſuppoſing CEF as before,

It will be, as radius : co-ſine of F : : tang. CF : tang. FE *(by the latter part of Theor.* 1.) that is, radius : ſine AB : : co-tang. BC : co-tang. A : : tang. A : tang. BC *(by Corol.* 5. *p.* 13.) *Q. E. D.*



COROL-

1 COROLLARY.



Hence it follows, that, in right-angled ſpherical triangles ABC, DBC, having the ſame perpendicular BC, the ſines of the baſes will be to each other, *inverſely*, as the tangents of the angles at the baſes:

For { radius : ſine AB : : tang. A : tang. BC }
ſince { radius : ſine DB : : tang. D : tang. BC }
we ſhall (*by reaſoning as in Cor.* 1. *Theor.* 1.) have
Sine AB : ſine DB : : tang. D : tang. A.

THEOREM V.

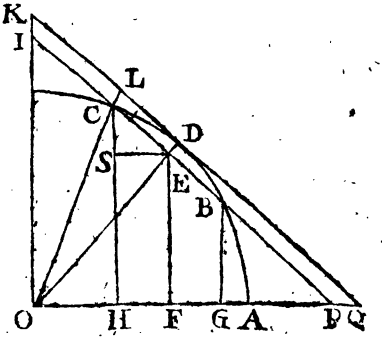
In any right-angled ſpherical triangle it will be, as radius is to the co-ſine of the hypothenuſe, ſo is the tangent of either angle to the co-tangent of the other angle.

For (CEF *being as in the laſt*) it will be, as radius : ſine CE : : tang. C : tang. EF (*by Theorem* 4.) that is, radius : co-ſine AC : : tang. C : co-tang. A. *Q. E. D.*

LEMMA.

As the ſum of the ſines of two unequal arches is to their difference, ſo is the tangent of half the ſum of thoſe arches to the tangent of half their difference: and, as the ſum of the co-ſines is to their difference, ſo is the co-tangent of half the ſum of the arches to the tangent of half the difference of the ſame arches.

For,



For, let AB and AC be the two propoſed arches, and let BG and CH be their ſines, and OG and OH their coſines: moreover, let the arch BC be equally divided in D, ſo that CD may be half the difference, and AD half the ſum, of AB and AC: let the radii OD and OC be drawn, and alſo the chord CB, meeting OE in E and OA (produced) in P; draw ES parallel to AO, meeting CH in S, and EF and OK perpendicular to AO, and let the latter meet EC (produced) in I; laſtly, draw QDK perpendicular to OD, meeting OA, OC and OI (produced) in Q, L and K.

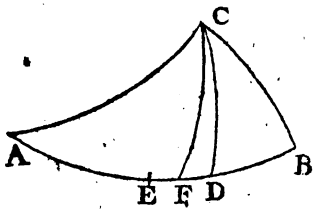
Becauſe CD = BD, it is manifeſt that OD is not only perpendicular to the chord BC, but biſects it in E; whence, alſo, EF biſects HG; and therefore CH + BG = 2EF, and CH — BG = 2CS; alſo OG + OH = 2OF, and OG — OH = 2HF: but 2EF (CH + BG) : 2CS (CH — BG) :: EF : CS :: EP : EC (*by* 14. 4.) :: DQ (the tangent of AD) : DL (the tangent of DC, *by* 20. 4.) And 2OF (OG + OH) : 2HF (OG — OH) :: EI : EC :: DK (the co-tang. of AD) : DL (the tang. DC). *Q. E. D.*

Theorem VI.

In any ſpherical triangle ABC *it will be, as the co-tangent of half the ſum of the two ſides is to the tangent of half their difference, ſo is the co-tangent of half the baſe to the tangent of the diſtance* (DE) *of the perpendicular from the middle of the baſe.*

Demon-

DEMONSTRATION.



Since co-sine AC : co-sine BC : : co-sine AD : co-sine BD (*by Cor. to Theor.* 2.) therefore, by composition and division, co-sine AC + co-sine BC : co-sine AC — co-sine BC : : co-sine AD + co-sine BD : co-sine AD — co-sine BD. But (*by the preceding lemma*) co-sine AC + co-sine BC : co-sine AC — co-sine BC : : co-tang. $\frac{AC+BC}{2}$: tang. $\frac{AC-BC}{2}$; and co-sine AD + co-sine BD : co-sine AD — con-sine BD : : co-tang. of AE $\left(\frac{AD+BD}{2}\right)$: tang. DE $\left(\frac{AD-BD}{2}\right)$; whence, by equality, co-tang. $\frac{AC+BC}{2}$: tang. $\frac{AC-BC}{2}$: : co-tang. AE : tang. DE.

COROLLARY.

Since the last proportion by permutation, becomes co-tang. $\frac{AC+BC}{2}$: co-tang. AE : : tang. $\frac{AC-BC}{2}$: tang. DE, and it is proved, in *p.* 13. that the tangents of any two arches are, inversely, as their co-tangents; it follows, therefore, that tang. AE : tang. $\frac{AC+BC}{2}$: : tang. $\frac{AC-BC}{2}$: tang. DE; or, that *the tangent of half the base, is to the tangent of half the sum of the sides, as the tangent of half the difference of the sides, to the tangent*

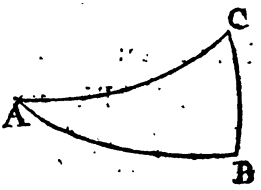
tangent of the diſtance of the perpendicular from the middle of the baſe.

THEOREM VII.

In any ſpherical triangle ABC, *it will be, as the co-tangent of half the ſum of the angles at the baſe, is to the tangent of half their difference, ſo is the tangent of half the vertical angle, to the tangent of the angle which the perpendicular* CD *makes with the line* CF *biſecting the vertical angle.* (See the preceding figure.)

DEMONSTRATION.

It will be (*by Corol. to Theor.* 3.) co-ſine A : co-ſine B : : ſine ACD : ſine BCD ; and therefore, co-ſine A + co-ſine B : co-ſine A — co-ſine B : : ſine ACD + ſine BCD : ſine ACD — ſine BCD. But (*by the lemma*) co-tang. $\frac{B+A}{2}$: tang. $\frac{B-A}{2}$: : (co-ſine A + co-ſine B : co-ſine A — co-ſine B : : ſine ACD + ſine BCD : ſine ACD — ſine BCD) tang. ACF : tang. DCF. *Q. E. D.*



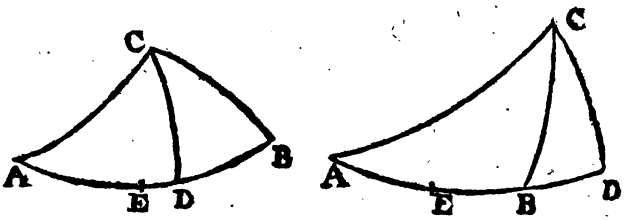
The ſolution of the caſes of right-angled ſpherical triangles.

Caſe.	Given.	Sought.	Solution.
1	The hyp. AC and one angle A	The oppoſite leg BC	As radius : ſine hyp. AC : : ſine A : ſine BC (*by the former part of Theor.* 1.)
2	The hyp. AC and one angle A	The adjacent leg AB	As radius : co-ſine of A : : tang. AC : tang. AB (*by the latter part of Theor.* 1.)
3	The hyp. AC and one angle A	The other angle C	As radius : co-ſine of AC : : tang. A : co-tang. C (*by Theor.* 5.)
4	The hyp. AC and one leg AB	The other leg BC	As co-ſine AB : radius : : co-ſine AC : co-ſine BC (*by Theor.* 2.)
5	The hyp. AC and one leg AB	The oppoſite angle C	As ſine AC : radius : : ſine AB : ſine C (*by the former part of Theor.* 1.)
6	The hyp. AC and one leg AB	The adjacent angle A	As tang. AC : tang. AB : radius : co-ſine A (*by Theor.* 1.)
7	One leg AB and the adjacent angle A	The other leg BC	As radius : ſine AB : : tangent A : tangent BC (*by Theor.* 4.)

Caſe.

Case.	Given	Sought	Solution.
8	One leg AB and the adjacent angle A	The oppoſite angle C	As radius : ſine A : : co-ſine of AB : co-ſine of C (*by Theor.* 3.)
9	One leg AB and the adjacent angle A	The hyp. AC	As co-ſine of A : radius : : tang. AB : tang. AC (*by Theor.* 1.)
10	One leg BC and the oppoſite angle A	The other leg AB	As tang. A : tang. BC : : radius : ſine AB (*by Theor.* 4.)
11	One leg BC and the oppoſite angle A	The adjacent angle C	As co-ſine BC : radius : : co-ſine of A : ſine C (*by Theor.* 3.)
12	One leg BC and the oppoſite angle A	The hyp. AC	As ſin. A : ſin. BC : : radius : ſine AC (*by Theor.* 1.)
13	Both legs AB and BC	The hyp. AC	As radius : co-ſine AB : : co-ſine BC : co-ſine AC (*by Theor.* 2.)
14	Both legs AB and BC	An angle, ſuppoſe A	As ſine AB : radius : : tang. BC : tang. A (*by Theor.* 4.)
15	Both angles A and C	A leg, ſuppoſe AB	As ſin. A : co ſine C : : radius : co-ſine AB (*by Theor.* 3.)
16	Both angles A and C	The hyp. AC	As tang. A : co-tang. C : : radius : co-ſine AC (*by Theor* 5.)

Note, *The* 10*th*, 11*th, and* 12*th caſes are ambiguous; ſince it cannot be determined, by the data, whether* AB, C, *and* AC, *be greater or leſs than* 90 *degrees each.*



The solution of the cases of oblique spherical triangles.

Case.	Given	Sought	Solution.
1	Two sides AC, BC and an angle A oppofite to one of them	The angle B oppofite to the other	As fine BC : fine A : : fine AC : fine B (*by Cor.* 1. *to Theor.* 1.) *Note*, This cafe is ambiguous when BC is leſs than AC; ſince it cannot be determined from the data whether B be acute or obtuſe.
2	Two ſides AC, BC and an angle A oppoſite to one of them	The included angle ACB	Upon AB produced (if need be) let fall the perpendicular CD; then (*by Theor.* 5.) rad. : co-fine AC : : tang. A : co-tang. ACD; but (*by Cor.* 2. *to Theor.* 1.) as tang. BC : tang. AC : : co-fine ACD : co-fine BCD. Whence ACB = ACD ± BCD is known.
3	Two ſides AC, BC and an angle oppoſite to one of them	The other ſide AB	As rad. : co-fine A : : tang. AC : tang. AD (*by Theor.* 1.) and (*by Cor. to Theor.* 2.) as co-fine AC : co-fine BC : : co-fine AD : co-fine BD. *Note*, This and the laſt cafe are both ambiguous when the firſt is ſo.
4	Two ſides AC, AB and the included angle A	The other ſide BC	As rad. : co-ſin. A : : tang. AC : tang. AD (*by Theor.* 1.) whence BD is alſo known; then (*by Corol. to Theor.* 2.) as co fine AD : co-fine BD : : co-fine AC : co-fine BC.
5	Two ſides AC, AB and the included angle A	Either of the other angles, ſuppoſe B	As rad. : co-fine A : : tang. AC : tang. AD (*by Theor.* 1.) whence BD is known; then (*by Cor. to Theor.* 4.) as fine BD : fine AD : : tang. A : tang. B.

Caſe.

Cafe.	Given	Sought	Solution.
6	Two angles A, ACB and the ſide AC betwixt them.	The other angle B	As rad. : co-ſine AC : : tang. A : co-tang. ACD (*by Theor.* 5.) whence BCD is alſo known; then (*by Cor. to Theor.* 3.) as ſine ACD : ſine BCD : : co-ſine A : co-ſine B.
7	Two angles A, ACB and the ſide AC betwixt them.	Either of the other ſides, ſuppoſe BC	As rad. : co-ſine AC : : tang. A : co-tang. ACD (*by Theor.* 5.) whence BCD is alſo known; then as co-ſine of BCD: co-ſine ACD : : tan. AC : tang. BC (*by Cor.* 2. *to Theor.* 1.)
8	Two angles A, B and a ſide AC oppoſite to one of them.	The ſide BC oppoſite the other	As ſine B : ſine AC : : ſine A : ſine BC (*by Cor.* 1. *to Theor.* 1.)
9	Two angles A, B and a ſide AC oppoſite to one of them.	The ſide AB betwixt them.	As rad. : co-ſine A : : tang. AC : tang. AD (*by Theor.* 1.) and as tan. B : tang. A : : ſine AD : ſine BD (*by Cor. to Theor.* 4.) whence AB is alſo known.
10	Two angles A, B and a ſide AC oppoſite to one of them.	The other angle ACB	As rad. : co-ſine AC : : tang. A : co-tang. ACD (*by Theor.* 5.) and as co-ſine A : co-ſine B : : ſine ACD : ſine BCD (*by Cor. to Theor.* 3.) whence ACB is alſo known.
11	All the three ſides AB, AC and BC	An angle, ſuppoſe A	As tang. $\frac{1}{2}$AB : tang. $\frac{AC + BC}{2}$: : tang. $\frac{AC - BC}{2}$: tang. DE, the diſtance of the perpendicular from the middle of the baſe (*by Cor. to Theor.* 6.) whence AD is known; then, as tang. AC : tang. AD : : rad. : co-ſine A (*by Theor.* 1.)
21	All the three angles A, B and ACB	A ſide, ſuppoſe AC	As co-tan. $\frac{ABC + A}{2}$: tan. $\frac{ABC - A}{2}$: : tang. $\frac{ACB}{2}$: tang. of the angle included by the perpendicular and a line biſecting the vertical angle; whence ACD is alſo known; then (*by Theor.* 5.) tang. A : co-tang. ACD : : rad. : co-ſine AC.

Note, In letting fall your perpendicular, let it always be from the end of a given ſide and oppoſite to a given angle.

Of the nature and conſtruction of Logarithms, with their application to the doctrine of Triangles.

AS the buſineſs of trigonometry is wonderfully facilitated by the application of logarithms; which are a ſet of artificial numbers, ſo proportioned among themſelves and adapted to the natural numbers 2, 3, 4, 5, &c. as to perform the ſame things by addition and ſubtraction, *only*, as theſe do by multiplication and diviſion: I ſhall here, for the ſake of the young beginner (for whom this ſmall tract is chiefly intended) add a few pages upon this ſubject. But, firſt of all, it will be neceſſary to premiſe ſomething, in general, with regard to the indices of a geometrical progreſſion, whereof logarithms are a particular ſpecies.

Let, therefore, $1, a, a^2, a^3, a^4, a^5, a^6, a^7$, &c. be a geometrical progreſſion whoſe firſt term is unity, and common ratio any given quantity a. Then it is manifeſt,

1. *That, the ſum of the indices of any two terms of the progreſſion is equal to the index of the product of thoſe terms.* Thus $2 + 3\ (5)$ is $=$ the index of $a^2 \times a^3$, or a^5; and $3 + 4\ (=7)$ is $=$ the index of $a^3 \times a^4$, or a^7. This is univerſally demonſtrated in p. 19. of my book of *Algebra*.

2. *That, the difference of the indices of any two terms of the progreſſion is equal to the index of the quotient of one of them divided by the other.* Thus $5-3$ is $=$ the index of $\frac{a^5}{a^3}$ or a^2. Which is only the converſe of the preceding article.

3. *That,*

3. *That, the product of the index of any term by a given number* (n) *is equal to the index of the power whoſe exponent is the ſaid number* (n). Thus 2×3 (6) is = the index of a^2 raiſed to the 3d power (or a^6). This is proved in p. 38, and alſo follows from article 1.

4. *That, the quotient of the index of any term of the progreſſion by a given number* (n) *is equal to the index of the root of that term defined by the ſame number* (n). Thus $\frac{6}{3}$ (2) is = the index of (a^2) the cube root of a^6. Which is only the converſe of the laſt article.

Theſe are the properties of the indices of a geometrical progreſſion; which being univerſally true, let the common ratio be now ſuppoſed indefinitely near to that of equality, or the exceſs of a above unity, indefinitely little; ſo that ſome term, or other, of the progreſſion 1, a, a^2, a^3, a^4, a^5, &c. may be equal to, or coincide with, each term of the ſeries of natural numbers, 2, 3, 4, 5, 6, 7, &c. *Then are the indices of thoſe terms called logarithms of the numbers to which the terms themſelves are equal.* Thus, if $a^m = 2$, and $a^n = 3$, then will m and n be logarithms of the numbers 2 and 3 reſpectively.

Hence it is evident, that what has been above ſpecified, in relation to the properties of the indices of powers, is equally true in the logarithms of numbers; ſince logarithms are nothing more than the indices of ſuch powers as agree in value with thoſe numbers. Thus, for inſtance, if the logarithms of 2 and 3 be denoted by m and n; that is, if $a^m = 2$, and $a^n = 3$, then will the logarithm of 6, (the product of 2 and 3) be equal to $m+n$ (agreeable to article 1); becauſe 2×3 (6) $= a^m \times a^n = a^{m+n}$.

But we muſt now obſerve, that there are various forms or ſpecies of logarithms; becauſe it is evident that what has been hitherto ſaid, in reſpect to the properties of indices, holds equally true in relation to any equimultiples, or like parts, of them; which have, manifeſtly, the ſame properties and proportions, with regard to each other, as the indices themſelves. But the moſt ſimple kind of all, is *Neiper's*, otherwiſe called the *hyperbolical*.

The hyperbolical logarithm of any number is the index, of that term of the logarithmic progreſſion agreeing with the propoſed number, multiplied by the exceſs of the common ratio above unity.

Thus, if e be an indefinite ſmall quantity, the hyperbolic logarithm of the natural number agreeing with any term $\overline{1+e}|^n$ of the logarithmic progreſſion 1, $\overline{1+e}|$, $\overline{1+e}|^2$, $\overline{1+e}|^3$, $\overline{1+e}|^4$, &c. will be expreſſed by ne.

Proposition I.

The hyperbolic logarithm (L) *of a number being given, to find the number itſelf, anſwering thereto.*

Let $\overline{1+e}|^n$ be that term of the logarithmic progreſſion 1, $\overline{1+e}|$, $\overline{1+e}|^2$, $\overline{1+e}|^3$, $\overline{1+e}|^4$, &c, which is equal to the required number (N). Then, becauſe $\overline{1+e}|^n$ is, univerſally, $= 1 + ne + n.\frac{n-1}{2}.e^2 + n.\frac{n-1}{2}.\frac{n-2}{2}e^3$ &c. we ſhall, alſo, have $1 + ne + n.\frac{n-1}{2}.e^2 + n.\frac{n-1}{2}.\frac{n-2}{3}.e^3$ &c. $=$ N. But, becauſe n (from the nature of logarithms) is here ſuppoſed indefinitely great, it is evident, firſt, that the numbers connected to it by the ſign —, may be rejected, as far as any aſſigned

number of terms being indefinitely ſmall in compariſon of n: it is alſo evident, that they may be rejected in all the reſt of the terms of the ſeries; becauſe theſe terms (by reaſon of the indefinite ſmallneſs of e) bear no aſſignable proportion to the preceding ones. Hence we have $1 + ne + \frac{n^2e^2}{2} + \frac{n^3e^3}{2.3} + \frac{n^4e^4}{2.3.4}$ &c. $= N$: but ne is $(=L)$ the hyperbolic logarithm of $\overline{1+e}|^n$ (or N) by what has been already ſpecified: therefore $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5}$ &c. $= N$. *Q. E. I.*

Prop. II.

To determine the hyperbolic logarithm (L) *of any given number* (N).

It appears from the preceding Prop. that $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3}$ &c. is $= N$: therefore, if $x + 1$ be put $= N$, we ſhall have $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4}$ &c. $= x$; and conſequently, by reverting the ſeries, $L = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$ &c. *Q. E. I.*

Otherwise.

Becauſe $\overline{1 + e}|^n = N$ (by the definition of logarithms) we ſhall have $1 + e = N^{\frac{1}{n}} = \overline{1 + x}|^m$; by putting $1 + x = N$, and $m = \frac{1}{n}$. Therefore, $\overline{1 + x}|^m$ being $= 1 + mx + m.\frac{m-1}{2}.x^2 + m.\frac{m-1}{\ }$

$\frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$ &c. we have $e = mx + m \cdot \frac{m-1}{2} \cdot x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$ &c. where, m being rejected in the factors $m-1$, $m-2$, $m-3$, &c. as indefinitely ſmall in compariſon of 1, 2, 3, &c. the equation will become $e = mx - \frac{mx^2}{2} + \frac{mx^3}{3} - \frac{mx^4}{4}$ &c. whence $\frac{e}{m} (= ne = L) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. the very ſame as before,

But this ſeries, tho' indeed the moſt eaſy and natural, is of little uſe in determining the logarithms of large numbers; ſince, in all ſuch caſes, it diverges, inſtead of converging. It will be proper, therefore, to give, here, the invention of other methods, which authors have had recourſe to, in order to obtain a ſeries that will always converge.

Firſt, then, let the number whoſe logarithm you would find be denoted by $\frac{1}{1-x}$; where it is manifeſt (however great that number may be) x will be always leſs than unity: moreover, let $\overline{1+e}|^n$ (as before) be the term of the logarithmic progreſſion agreeing with the propoſed number, or, which is the ſame, let $\overline{1+e}|^n = \frac{1}{1-x}$: then, (by taking the root on both ſides) we ſhall have $1+e = \frac{1}{\overline{1-x}|^{\frac{1}{n}}} = \overline{1-x}|^{-\frac{1}{n}} = \overline{1-x}|^m$

(by

(by making $m = -\frac{1}{n}$) $= 1 - mx + m.\frac{m-1}{2}$ $x^2 - m.\frac{m-1}{2}.\frac{m-2}{3}.x^3$ &c. where m being rejected in the factors $m-1$, $m-2$, &c. (as before) our equation will become $1+e = 1 - mx - \frac{mx^2}{2} - \frac{mx^3}{3}$ &c. whence $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ &c. $= \frac{e}{-m} = ne =$ the hyperbolic logarithm of $\frac{1}{1-x}$. Which ſeries, it is manifeſt, will always converge, let the value of $\frac{1}{1-x}$ be ever ſo great; becauſe x will be always leſs than unity.

But it is further obſervable that this ſeries has exactly the ſame form (except in its ſigns) with that above for the logarithm of $1+x$; and that, if both of them be added together, the ſeries $2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}$ &c. thence ariſing, will be more ſimple than either of them; ſince one half of the terms will be intirely deſtroyed thereby. Therefore, becauſe the ſum of the logarithms of any two numbers is equal to the logarithm of the product of thoſe numbers, (*ſee Article* 1.) it is manifeſt that $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$ &c. will truly expreſs the logarithm of $\frac{1+x}{1-x}$, or $\frac{1-x}{1} \times \overline{1+x}$. Which ſeries converges, ſtill, faſter than $x + \frac{x^2}{2} + \frac{x^3}{3}$ &c. not only becauſe the even powers are here deſtroyed, but becauſe x, in finding the logarithm of any given number (N), will have a leſs value.

But

But now, to determine what this value muſt be, make $\frac{1+x}{1-x} = N$, and then x will be found $= \frac{N-1}{N+1}$; but if the quantity propoſed be a fraction $\left(\frac{P}{Q}\right)$, inſtead of a whole number, make $\frac{P}{Q} = \frac{1+x}{1-x}$, and you will have $x = \frac{P-Q}{P+Q}$: either of which values, ſubſtituted in the foregoing ſeries $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$ &c. will give the hyperbolic logarithm of the reſpective number.

Example. Let it be propoſed to find the hyperbolic logarithm of the number 2.

Here x being $= \frac{2-1}{2+1} = \frac{1}{3}$, and $x^2 = \frac{1}{9}$; we ſhall have

$x \left(= \frac{1}{3}\right) = ,333333333$ &c.
$x^3 \left(= \frac{1}{9}x\right) = ,037037037$ &c.
$x^5 \left(= \frac{1}{9}x^3\right) = ,004115226$ &c.
$x^7 \left(= \frac{1}{9}x^5\right) = ,000457247$ &c.
$x^9 \left(= \frac{1}{9}x^7\right) = ,000050805$ &c.
$x^{11} \left(= \frac{1}{9}x^9\right) = ,000005645$ &c.
$x^{13} \left(= \frac{1}{9}x^{11}\right) = ,000000627$ &c.
$x^{15} \left(= \frac{1}{9}x^{13}\right) = ,000000069$ &c.
&c. &c.

Which values being reſpectively divided by the numbers, 1, 3, 5, 7, 9, &c. and the ſeveral quotients added together, *(ſee the general ſeries)* we ſhall have ,346573590 &c. whoſe double, being ,693147180 &c. is the hyperbolical logarithm of the number 2.

After

After the very ſame manner the hyperbolic logarithm of any other number may be determined; but, as the ſeries converges, ſlower and ſlower, the higher we go, it is uſual, in computing of tables, to derive the logarithms we would find, by help of others already known; for which there are various methods; but the following is the moſt commodious and ſimple, that has occurred to me, eſpecially, when a great degree of accuracy is required.

It is thus. Let a, b and c denote any three numbers in arithmetical progreſſion, whoſe common difference is unity; then, a being $=b-1$ and $c=b+1$, we ſhall have $ac=b^2-1$, and conſequently $\frac{b^2}{ac}=\frac{ac+1}{ac}$. Whence, by the nature of logarithms, we likewiſe have $2\log. b-\log. a-\log. c=\log.\frac{ac+1}{ac}$: but the logarithm of $\frac{ac+1}{ac}$, by putting $\frac{1}{2ac+1}=x$, will be $\frac{2x}{1}+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}$ &c. (by what has been already ſhewn): which being denoted by S, we ſhall

have $\left\{\begin{array}{l}\log. b=\frac{1}{2}\log. a+\frac{1}{2}\log. c+\frac{1}{2}S.\\ \log. a=2\log. b-\log. c-S.\\ \log. c=2\log. b-\log. a-S.\end{array}\right\}$

As an example hereof, let it be propoſed to find the hyperbolic logarithm of 3.

Then, the hyperbolic logarithm of 2 being already found $=,693147180$ &c. that of 4, which is the double thereof, will alſo be known. Therefore, taking $a=2$, $b=3$, and $c=4$, we ſhall, in this

this caſe, have $x\left(\frac{1}{2ac+1}\right) = \frac{1}{171}$, $x^2 = \frac{1}{289}$; whence

$x\ (= \frac{1}{17}) = ,058823529$ &c.
$x^3\ (= \frac{x}{289}) = ,000203542$ &c.
$x^5\ (= \frac{x^3}{289}) = ,000000704$ &c.
&c. &c.

Therefore $\frac{1}{2}S(x+\frac{x^3}{3}+\frac{x^5}{5}$ &c.$) = ,058891517$ &c. and conſequently hyp. log. 3. $\left(\frac{\text{hyp. log. } 2+\text{hyp. log. } 4}{2}\right.$ $\left. + \frac{1}{2}S\right) = 1,098612288$ &c.

2. Let the hyperbolic logarithm of 10 be required.

The logarithms of 8 and 9 being given, from thoſe of 2 and 3 (already found), a may, here, be $=8$, $b=9$ and $c=10$; and then $x\left(\frac{1}{2ac+1}\right)$ being $= \frac{1}{161}$, we ſhall have $\frac{1}{2}S\left(x+\frac{x^3}{3}+\frac{x^5}{5}\right.$ &c.$) =$,006211180 &c. + ,000000079 &c. &c. = ,006211259 &c.

And therefore log. 10 ($2\ log.\ 9 - log.\ 8 - S$) $= 2,302585092$ &c.

Hitherto we have had regard to logarithms of the hyperbolic kind: but thoſe of any other kind may be derived from theſe, by, *barely*, multiplying by the proper multiplicator, or modulus.

Thus, in the *Brigean* (or common) form, where an unit is aſſumed for the logarithm of 10, the logarithm of any number will be found, by multiplying

tiplying the hyperbolic logarithm of the ſame number by the fraction ,434294481 &c. which is the proper modulus of this form.

For, ſince the logarithms of all forms preſerve the ſame proportion with reſpect to each other, it will be, as 2,302585092 &c. the hyperbolic log. of 10 (above found) is to (H) the hyperbolic logarithm of any other number, ſo is 1, the common logarithm of 10, to $\left(\frac{H}{2,302585092\&c.}\right)$ $H \times ,434294481$ &c. the common logarithm of the ſame number.

But (to avoid a tedious multiplication, which will always be required when a great degree of accuracy is inſiſted on) the beſt way to find the logarithms of this form is from the ſeries $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$ &c. $\times ,434294481$ &c. which expreſſes the common logarithm of $\frac{1+x}{1-x}$ (by what has been already ſhewn), and which, by making $R = ,868588963$ &c. will ſtand more commodiouſly thus, $Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5} + \frac{Rx^7}{7}$ &c.

For an example hereof, let the common logarithm of 7 be required: in which caſe the logarithms of 8 and 9 being known, from thoſe of 2 and 3), we ſhall have log. $7 = 2$ log. $8 -$ log. $9 - S$ (*by the Theor.*), S being $= Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5}$ &c. $\left(=\text{the common log of } \frac{64}{63}\right)$ and $x \left(=\frac{64-63}{64+63}\right) = \frac{1}{127}$: whence $\left(x^2 \text{ being} = \frac{1}{16129}\right)$ we ſhall have

Rx

$$Rx\ \left(=\frac{,8685\ \&c.}{127}\right) = ,006839283\ \&c.$$

$$Rx^3\ \left(=\frac{Rx}{16129}\right) = ,000000424\ \&c.$$

$$Rx^5\ \left(=\frac{Rx^3}{16129}\right) = ,00000000002\ \&c.$$

&c. &c.

Consequently $S\ (Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5}\ \&c.) =$,006839424 &c. and 2 log. 8 — log. 9 — S = ,845098040 &c. = the common logarithm of 7 required. But the same conclusion may be brought out by fewer terms of the series, if the logarithms of the three first primes 2, 3 and 5 be supposed known; because those of 48 and 50 (which are composed of them) will likewise be known; from whence the logarithm of 7 $\left(= \frac{1}{4}\text{ log. } 49 = \frac{\textit{log. } 48. + \textit{log. } 50 + S}{4}\right)$ will come out =,845098040 &c. (*as before*) which value will be true to 11 places of figures by taking the first term of the series, *only*.

Again, let the common logarithm of the next prime number, which is 11, be required. Here a may be taken = 10, b = 11, and c = 12; but, fewer terms of the series will suffice, if other three numbers, composed of 11 and the inferior primes, be taken, whereof the common difference is an unit. Thus, because $98 = 2 \times 7 \times 7$, $99 = 3 \times 3 \times 11\ (9 \times 11)$, and $100 = 2 \times 2 \times 5 \times 5$ (or 10×10), let there be taken $a = 98$, $b = 99$, and $c = 100$; and then, by the first term of the series only, the log. of 99 will be found true to 14 places; whence that of 11 (log. 99—log. 9.) is also known.

But

But notwithſtanding all theſe artifices and compendiums, a method (ſimilar to that in page 18.) for finding the logarithms of large numbers, one from another, by addition and ſubtraction, only, ſtill ſeems wanting in the calculation of tables; I ſhall, therefore, here ſubjoin ſuch a method.

1. Let A, B and C denote any three numbers in arithmetical progreſſion, not leſs than 10000 each, whereof the common difference is 100.

2. From twice the logarithm of B, ſubtract the ſum of the logarithms of A and C, and let the remainder be divided by 10000.

3. Multiply the quotient by 49,5, and to the product add $\frac{1}{100}$ part of the difference of the logarithms of A and B; then the ſum will be the exceſs of the logarithm of A + 1 above that of A.

4. From this exceſs let the quotient (found by Rule 2.) be continually ſubtracted, that is, firſt from the exceſs itſelf, then from the remainder, then from the next remainder, &c. &c.

5. To the logarithm of A add the ſaid exceſs, and to the ſum add the firſt of the remainders; to the laſt ſum add the next remainder, &c. &c. then the ſeveral ſums, thus ariſing, will exhibit the logarithms of A + 1, A + 2, A + 3, &c. reſpectively.

Thus, let it be propoſed to find the logarithms of all the whole numbers between 17900 and 18100; thoſe of the two extremes 17900 and 18100, and that of the mean (18000) being given.

Then the logarithm of $\left\{\begin{matrix} A \\ B \\ C \end{matrix}\right\}$ being $\left\{\begin{matrix} 4,252853031 \\ 4,255272505 \\ 4,257678575 \end{matrix}\right\}$ we shall have $\frac{2 \text{ log. } B - \text{log. } A - \text{log. } C}{10000} =$,000000001134 (*see Rule* 2.) which multiplied by 49,5, and the product added to $\frac{\text{log. } B - \text{log. } A}{100}$ gives ,000024261107 for the excefs of the logarithm of A + 1 above that of A (*by Rule* 3.) From whence the work, being continued according to Rule 4 and 5, will ftand as follows:

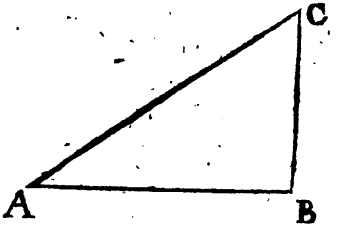
,000024	26107	excefs.	4,25	2853031	log. 17900
	134			2426107	excefs.
	25973	1ft rem.		287729207	log. 17901
	134			2425973	
	25839	2d rem.		290155180	log. 17902
	134			2425839	
	25705	3d rem.		292581019	log. 17903
	134			2425705	
	25571	4th rem.		295006724	log. 17904
	134			2425571	
	25437	5th rem.		297432295	log. 17905
	134			2425437	
	25303	6th rem.		299857732	log. 17906
	134			2425303	
	25169	th rem.		302283035	log. 17907
	134			2425169	
	25035	th rem.		304708204	log. 17908
	134			2425035	
	24901	9th rem.		307133239	log. 17909
	134			2424901	
	24767	10th rem.		309558140	log. 17910
	&c.	&c.		&c.	&c.

Note,

* *Note,* The logarithms found according to this method, in numbers between 10000 and 20000, are true to 8 or 9 places of figures: thoſe of numbers between 20000 and 50000 err only in the 9th or 10th place; and thoſe of above 50000 are true to 10 places, at leaſt.

Having explained the manner of conſtructing a table of logarithms, and that by various methods, I now come to ſhew the uſe of ſuch a table in the buſineſs of trigonometry.

Firſt, in the right-angled plane triangle ABC, let there be given the hypothenuſe AC = 17910 feet, and the angle A = 35° 20′; to find the perpendicular BC and the baſe AB.



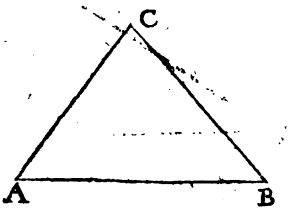
Here, becauſe radius : ſine 35° 20′ :: 17910 : BC (*by Theor.* 2. *p.* 6.) we have $BC = \frac{\text{ſine } 35° 20' \times 17910}{\text{radius}}$; therefore, becauſe the addition and ſubtraction of logarithms anſwers to the multiplication and diviſion of the natural numbers (*ſee p.* 38, 39.) we have *log.* BC = *log. ſine* 35° 20′ + *log.* 17910 — *log. radius.*

But, by the tables of artificial, or logarithmic, ſines *, the log. ſine of 35° 20′ will appear to be 9,7621775; to which add 4,2530956, the log. of 17910, and from the ſum (14,0152731.) take 10, the log. of radius, and there reſults 4,0152731 = the log. of BC; which, in the tables, anſwers to 10358, the length of BC required.

* *A table of artificial ſines is nothing more than a table of the logarithms of the numbers expreſſing the natural ſines, to the radius* 10000000000; *whoſe logarithm is* 10.

Again, for AB, it will be, as radius : ſine of C (54° 40′) : : AC (17910) : AB (*by Theorem* 2.) Whence, by adding the logarithms of the ſecond and third terms together and ſubtracting that of the firſt (as above), we have AB = 14611. See the operation.

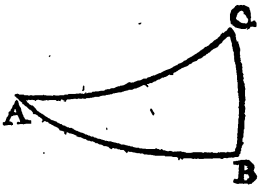
Log. radius - - - - -	10,
Log. ſine C (54° 40′)	9,9115844
Log. AC (17910) - -	4,2530956
Log. AB = 14611 -	4,1646800



Moreover, in the oblique plane triangle ABC, let there be given AB = 75, AC = 60, and the included angle A = 48°; to find the other two angles. Then (*by Theorem* 5.) it will be,

As AB + AC (135)	its log.	2,1303338
is to AB — AC (15)	its log.	1,1760913
ſo is T. $\frac{C+B}{2}$ (66°)	- - - -	10,3514169
		11,5275082
to the T. $\frac{C-B}{2}$ = 14° 00′		9,3971744

Which 14° added to 66°, the half ſum of the angles C and B, gives the greater C = 80°; and ſubtracted therefrom, leaves the leſſer B = 52°.



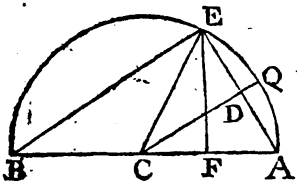
Laſtly, in the right-angled ſpherical triangle ABC, let there be given the hypothenuſe AC = 60°, and the angle A = 23° 29′; to find the baſe and perpendicular. Then (*by Theor.* 1. *p.* 25.) the operation will be as follows:

Log. radius - - - - - - - - - -	10,
Log. ſine A (23° 29′) - - - -	9,6004090
Log. ſine AC (60°) - - - - -	9,9375306
Log. ſine BC = 20° 11′ - - -	9,5379396
Alſo, log. radius - - - - - - -	10,
Log. Co-ſ. A (23° 29′) - - -	9,9624527
Log. T. AC (60°) - - - - -	10,2385606
Log. T. AB = 57° 49′ - - -	10,2010133

Having exhibited the manner of reſolving all the common caſes of plane and ſpherical triangles, both by logarithms and otherwiſe; I ſhall here ſubjoin a few propoſitions for the ſolution of the more difficult caſes which ſometimes occur; when, inſtead of the ſides and angles themſelves, their ſums, or differences, &c. are given.

Proposition I.

The ſine, co-ſine, or verſed ſine of an arch being given, to find the ſine and co-ſine, &c. of half that arch.

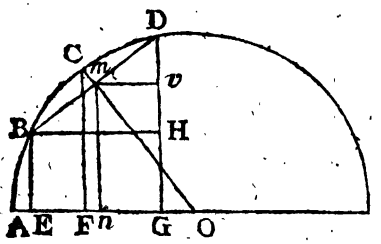


From the two extremes of the diameter AB, let the chords AE and BE be drawn, and let the radius CQ biſect AE, perpendicularly, in D (*Vid.* 1. 3.); then will AD be the ſine, and CD the co-ſine, of the angle ACD, or $\frac{1}{2}$ACE.

But $4AD^2 = AE^2$ (*by Cor.* 1. to 6. 2.) $= AB \times AF$ (*by Cor. to* 19. 4.) $= 2AC \times AF$; whence $AD^2 = \frac{1}{2}AC \times AF$: alſo $4CD^2 = BE^2 = AB \times BF = 2AC \times BF$; whence $CD^2 = \frac{1}{2}AC \times BF$. *From which it appears, that the ſquare of the ſine of half any arch, or angle, is equal to a rectangle under half the radius and the verſed ſine of the whole; and that the ſquare of its co-ſine is equal to a rectangle under half the radius and the verſed ſine of the ſupplement of the whole arch, or angle.*

Prop. II.

The ſines and co-ſines of two arches being given, to find the ſines, and the co-ſines, of the ſum and difference of theſe arches.



Let AC and CD (=BC) be the two propoſed arches; let CF and OF be the ſine and co-ſine of the greater AC, and let *m*D (B*m*) and O*m*, be thoſe of the leſſer CD (or BC): moreover, let DG and OG be the ſine and co-ſine of the ſum AD; and BE and OE, thoſe of the difference AB. Draw *mn* parallel to CF, meeting AO in *n*; alſo draw *mv*

and BH parallel to AO, meeting GD in v and H: then it is plain, because $Dm = Bm$, that Dv is $= Hv$, and $mv = nG = En$; and that the triangles OCF, Omn and mDv are similar; whence we have the following proportions,

$$\left.\begin{array}{l} OC:Om::CF:mn \\ OC:OF::Dm:Dv \\ OC:OF::Om:On \\ OC:CF::Dm:mv \end{array}\right\} \text{whence} \left\{\begin{array}{l} mn \times OC = Om \times CF. \\ Dv \times OC = Dm \times OF. \\ On \times OC = Om \times OF. \\ mv \times OC = Dm \times CF. \end{array}\right.$$

Now, by adding the two first of these equations together, we have $\overline{mn + Dv} \times OC$ (DG $\times$ OC) $=$ $Om \times CF + Dm \times OF$; whence DG is known. Moreover, by taking the latter from the former, we get $\overline{mn - Dv} \times OC$ (BE $\times$ OC) $= Om \times CF -$ $Dm \times OF$; whence BE is known.

In like manner, by adding the third and fourth equations together, we have $\overline{On + mv} \times OC$ (OE $\times$ OC) $= Om \times OF + Dm \times CF$; and, by subtracting the latter from the former, we have $\overline{On - mv} \times$ OC (OG $\times$ OC) $= Om \times OF - Dm \times CF$; whence OE and OG are also known. *Q. E. I.*

Corollary I.

Hence, if the sines of two arches be denoted by S and s; their co-sines by C and c; and radius by R; then will

the sine of their sum $= \frac{Sc + sC}{R}$,

the sine of their difference $= \frac{Sc - sC}{R}$,

the co-sine of their sum $= \frac{Cc - Ss}{R}$,

the co-sine of their difference $= \frac{Cc + Ss}{R}$:

Corollary II.

Hence, the ſine of the double of either arch (when they are equal) will be $= \frac{2CS}{R}$, and its co-ſine $= \frac{C^2 - S^2}{R}$: whence it appears, that *the ſine of the double of any arch, is equal to twice the rectangle of the ſine and co-ſine of the ſingle arch, divided by radius; and that its co-ſine is equal to the difference of the ſquares of the ſine and co-ſine of the ſingle arch, alſo, divided by radius.*

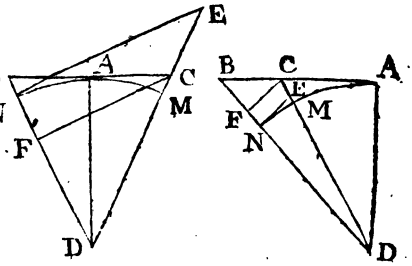
Corollary III.

Moreover, becauſe $Dm \times CF = OC \times mv$ ($\frac{1}{2}OC \times EG = \frac{1}{2}OC \times \overline{OE - OG}$); and $Om \times OF = OC \times On$ ($\frac{1}{2}OC \times 2On = \frac{1}{2}OC \times \overline{OE + OG}$), it follows, *that the rectangle of the ſines of any two arches* (AC, CD (BC) *is equal to a rectangle under half the radius, and the difference of the co-ſines, of the ſum and difference of thoſe arches; and that the rectangle of their co-ſines is equal to a rectangle under half the radius, and the ſum of the co-ſines, of the ſum and difference of the ſame arches.*

Prop. III.

The tangents of two arches being given, to find the tangents of the ſum, and difference, of thoſe arches.

Let



Let AN and AM be the two arches, and AB and AC their tangents; alſo let NE be the tangent of their ſum, in the firſt caſe, and the tangent of their difference, in the ſecond, and let CF, perpendicular to the radius DN, be drawn: then, becauſe of the equiangular triangles BAD and BFC, we ſhall have

$$\left.\begin{matrix} BD \times CF = DA \times BC \\ BD \times BF = BA \times BC \end{matrix}\right\} \text{by } 24.\ 3.$$

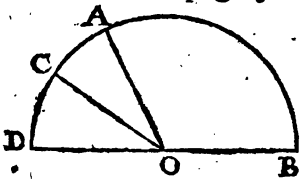
Take each of the laſt equal quantities from BD^2, and there will remain $BD^2 - BD \times BF\ (BD \times DF) = BD^2 - BA \times BC$: now $BD \times DF\ (BD^2 - BA \times BC) : BD \times CF\ (DA \times BC) :: DF : CF :: DN\ (DA) : NE :: DA^2 : A \times NE$; whence, alternately, $BD^2 - BA \times BC : DA^2\ (:: DA \times BC : DA \times NE) :: BC : NE$. But the firſt term, of this proportion, becauſe $BD^2 = DA^2 + BA^2$, will alſo be expreſſed by $DA^2 + BA^2 - BA \times BC$, or by $DA^2 \pm BA^2 - BA \times \overline{AB \pm AC}$; or, laſtly, by, $DA^2 + BA \times AC$: therefore, the three firſt terms of the proportion being known, the fourth NE will likewiſe be known. *Q. E. I.*

Corollary.

Hence, if radius be ſuppoſed unity, and the tangents of two arches be denoted by T and t, it follows, that the tangent of their ſum will be $= \frac{T + t}{1 - tT}$, and the tangent of their difference $= \frac{T - t}{1 + tT}$.

But,

But, it will be proper to take notice here (once for all) that, if in theſe, or any other theorems, the tangent, ſecant, co-ſine. co-tangent, &c. of an arch greater than 90 degrees be concerned; then, inſtead thereof, the tangent, ſecant, co-ſine, &c. of an arch, as much below 90 degrees, is to be taken, with a negative ſign; according to the obſervation in page 5.

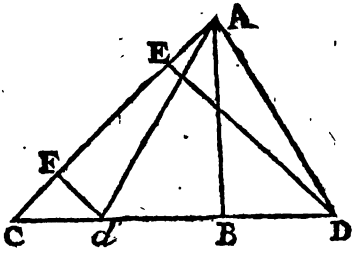


Thus, for inſtance, let BA be an arch greater than 90°, and let the tangent of the ſum of AB and AC be required; ſuppoſing T to repreſent the tangent of AD (the ſupplement of AB) and t the tangent of AC: then, by writing $-T$ inſtead of T, in the firſt of the foregoing theorems, we ſhall have tang. of $BC = \frac{-T+t}{1-t\times-T} = \frac{-T+t}{1+tT}$, and therefore tang. DC (—tang. BC) $= \frac{T-t}{1+tT}$; which is the very theorem demonſtrated in the 2d caſe.

Prop. IV.

As the ſum of the tangents of any two angles BAC, BAD, *is to their difference; ſo is the ſine of the ſum of thoſe angles, to the ſine of their difference.*

Let



Let BC and BD be the two propoſed tangents, to the radius AB; take B*d* = BD, join A*d*, and draw DE and *d*F perpendicular to AC. It is manifeſt, becauſe B*d* = BD, that A*d* = AD, and *d*AB = DAB, and, conſequently, that CA*d* is the difference of the two angles BAC and BAD.

Now, by reaſon of the ſimilar triangles CDE and C*d*F, it will be, CD (CB+BD) : C*d* (CB—BD) : : DE : *d*F; but DE and *d*F are ſines of DAE and *d*AF to the equal radii AD and A*d*: *whence the truth of the propoſition is manifeſt.*

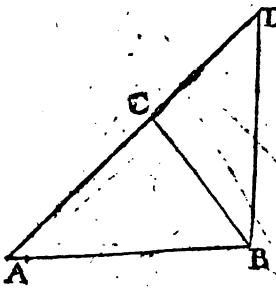
Corollary.

Hence it alſo appears, that the baſe (CD) of a plane triangle, is to (C*d*) the difference of its two ſegments (made by letting fall a perpendicular), as the ſine of the angle (CAD) at the vertex, to the ſine of the difference of the angles at the baſe.

Prop. V.

In any plane triangle ABC, *it will be, as the ſum of the two ſides* plus *the baſe, is to the ſum of the two ſides* minus *the baſe, ſo is the co-tangent of half either angle at the baſe, to the tangent of half the other angle at the baſe.*

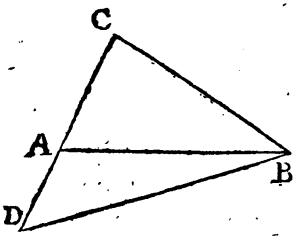
In



In AC produced, take CD = BC, and let BD be drawn: then (*by Theor.* 5. *p.* 8.) it will be, AD + AB : AD — AB : : tang. $\frac{ABD + D}{2}$ ($\frac{180° - A}{2}$ = 90° — ½A) : tang. $\frac{ABD - D}{2}$ ($\frac{ABD - CBD}{2}$ = $\frac{ABC}{2}$), that is, AC + BC + AB : AC + BC — AB : : co-tang. ½A : tang. ½ABC. *Q. E. D.*

Prop. VI.

In any plane triangle ABC, *it will be, as the base* plus *the difference of the two sides, is to the base* minus *the same difference, so is the tangent of half the greater angle at the base, to the tangent of half the lesser.*

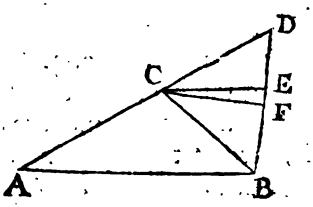


In the lesser side CA, produced, take CD = CB, so that AD may be the difference of the two sides; and let BD be drawn: then it is manifest that the angle CBD will be equal to D: but (*by Theor.* 5. *p.* 8.) AB + AD : AB — AD : : tangent $\frac{D + DBA}{2} = \frac{CAB}{2}$ (*by* 9. 1.) : tangent $\frac{D - DBA}{2} = \frac{CBD - DBA}{2} = \frac{CBA}{2}$. *Q. E. D.*

Prop.

PROP. VII.

As the base of any plane triangle ABC, *is to the sum of the two sides, so is the sine of half the vertical angle, to the co-sine of half the difference of the angles at the base.*



In AC, produced, take CD = CB; join B, D, and draw CE parallel to AB, and CF perpendicular to BD.

Since CD = CB, therefore is the angle D = DBC (*by* 12. 1.) and, consequently, half the vertical angle $ACB = \frac{D + CBD}{2}$ (*by* 9. 1.) = D.

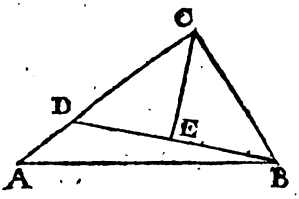
Moreover, seeing DCB is = the sum of the angles A and CBA, at the base (*by* 9. 1.) it is evident that BCF (or DCF) is equal to half that sum; and, therefore, as ECF is the excess of the greater ABC = BCE (*by* 7. 1.) above the half sum (BCF), it must, manifestly, be equal to half the difference of the same angles A and CBA.

But (*by Theor.* 3.) AB : AD (AC + BC) : : sine D ($\frac{1}{2}$ACB) : sine ABD = sine CED (*by Cor.* 1. *to* 7. 1.) = sine FEC = co-sine ECF. *Q. E. D.*

PROP. VII.

As the base of any plane triangle ABC, *is to the difference of the two sides, so is the co-sine of half the vertical angle, to the sine of half the difference of the angles at the base.*

In

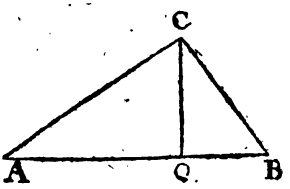


In the greater ſide CA let there be taken CD = CB, and let BD be drawn, and likewiſe, CE, perpendicular to BD. It is manifeſt, becauſe CD = CB, that CDB and CBD are equal to one another, and that each of them is alſo equal to half the ſum of the angles CBA and A at the baſe (*by Cor.* 2. *to* 10. 1.); therefore ABD, being the exceſs of the greater CBA above the half ſum, muſt conſequently be equal to half the difference of the ſame angles.

But (*by Theor.* 3.) AB : AD (AC—BC) : : ſine D (co-ſine DCE, or $\frac{1}{2}$C) : ſine ABD. *Q. E. D.*

Prop. IX.

As the difference of the two ſides AC, BC, *of a plane triangle, is to the difference of the ſegments of the baſe* AQ, BQ *(made by letting fall a perpendicular from the vertex), ſo is the ſine of half the vertical angle, to the co-ſine of half the difference of the angles at the baſe.*



For, AC—BC : AQ—BQ : : AB : AC + BC (*by* 9. 2. *and* 10. 4.) : : ſine $\frac{ACB}{2}$: co-ſine $\frac{B-A}{2}$ (*by Prop.* 7.) *Q. E. D.*

Prop. X.

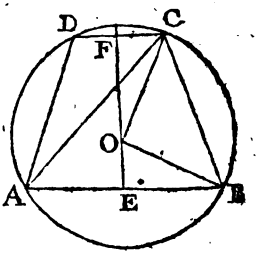
As the ſum of the two ſides of a plane triangle, is to the difference of the ſegment of the baſe (ſee the preceding figure), ſo is the co-ſine of half the vertical angle, to the ſine of half the difference of the angles at the baſe.

For,

For, AC+BC : AQ—BQ : : AB : AC—BC (*by* 9. 2. *and* 10. 4.) : : co-fine of $\frac{ACB}{2}$: fine of $\frac{B-A}{2}$ (*by Prop.* 8.) *Q. E. D.*

Prop. XI.

As the tangent of the vertical angle C *of a plane triangle* ABC, *is to radius, ſo is half the baſe* AB *to a fourth proportional; and as half the baſe is to the exceſs of the perpendicular above the ſaid fourth-proportional, ſo is the fine of the vertical angle, to the co-fine of the difference of the angles at the baſe.*



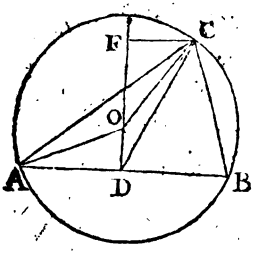
Let ABCD be a circle deſcribed about the triangle, and from O, the center thereof, let OB and OC be drawn; moreover, draw CD parallel to BA, meeting the periphery in D, and EOF, perpendicular to AB, meeting DC in E. Then it is evident, that EF will be equal to the perpendicular height of the triangle, EOB equal to the vertical angle ACB, and FOC (= DAC) equal to the difference of the angles (ABC and BAC) at the baſe.

But (*by Theor.* 2.) as tang. EOB (ACB) : radius : : EB ($\frac{1}{2}$AB) : EO; moreover, as EB : OF (EF — EO) : : fine EOB (ACB) : fine OCF, or co-fine of FOC. *Q. E. D.*

Prop.

PROP. XII.

As the tangent of the vertical angle, of a plane triangle ABC, *is to radius, ſo is the baſe* AB *to a fourth-proportional; and, as the ſaid fourth proportional, is to the ſum of the ſemi-baſe and the line* CD *biſecting the baſe, ſo is the difference of theſe two, to the perpendicular height of the triangle.*

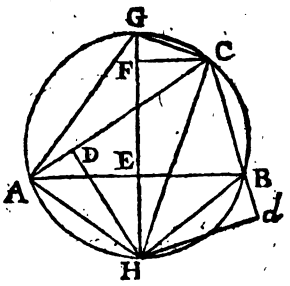


Let a circle be deſcribed about the triangle, and from O, the center thereof, let OA, OC and OD be drawn; alſo let CF, parallel to AB, be drawn, meeting DO, produced (if need be) in F. It is evident that DF will be perpendicular to AB, and equal to the height of the triangle. But $DC^2 = OC^2 + OD^2 + 2OD \times OF$ (*by* 11. 2.) $= OA^2 + OD^2 + 2OD \times \overline{DF - OD} = OA^2 - OD^2 + 2OD \times DF = AD^2 + 2OD \times DF$ (*by* 7. 2.); whence, by taking away AD^2 from the firſt and laſt of theſe equal quantities, we have $DC^2 - AD^2 = 2OD \times DF$; or $\overline{DC + AD} \times \overline{DC - AD} = 2OD \times DF$ (*by* 7. 2.) and therefore $2OD : DC + AD :: DC - AD : DF$; but (*by Theor.* 2.) as the tang. AOD = ACB (*by* 10. 3.) : to radius (:: AD : OD) : : AB : 2OD. *Q. E. D.*

PROP. XIII.

As twice the rectangle under the baſe and perpendicular of a plane triangle ABC, *is to the rectangle under the ſum, and difference, of the baſe and ſum of the two ſides; ſo is radius, to the co-tangent of half the vertical angle.*

Let



Let HG, perpendicular to AB, be the diameter of a circle described about the triangle; and let HD and H*d* be perpendicular to the two sides of the triangle; also let CF be parallel to AB, and let HA, HB and HC be drawn.

Since the diameter HG is perpendicular to AB, therefore is AE = BE (*by* 2. 3.) AH = BH, and the angle ACH = BCH (*by* 12. 3.); whence, also, CD = C*d*, and HD = H*d* (*by* 15. 1.) Therefore, the right-angled triangles HAD and HB*d*, having AH = HB and HD = H*d*, have, likewise, AD = B*d* (*by* 15. 1.) From whence it is manifest, that CD will be equal to half the sum, and AD equal to half the difference, of the two sides of the triangle. Moreover, because of the similar triangles AEH and HCD, it will be, $AE^2 : CD^2 :: HA^2 = HE \times HG$ (*by Cor. to* 19. 4.) : HC^2 (HF × HG) :: HE : HF (*by* 7. 4.) Whence, *by division*, &c. $AE^2 : CD^2 - AE^2 :: HE : EF :: HE \times AE : EF \times AE$; therefore, *by inversion and alternation*, $EF \times AE : CD^2 - AE^2 (CD + AE \times CD - AE) :: HE \times AE ; AE^2$. HE : AE :: radius : co-tang. EAH (ACH, *by Theor.* 2.): *whence the truth of the proposition is manifest.*

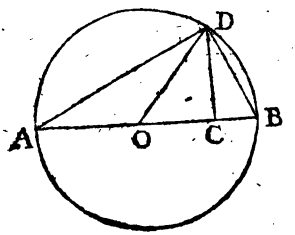
Prop. XIV.

As twice the rectangle of the base and perpendicular of a plane triangle ABC, *is to the rectangle under the sum, and difference, of the base and the difference of the two sides; so is radius, to the tangent of half the vertical angle.*

Let the preceding figure and conſtruction be retained, and let AG and CG be drawn. The triangles AHD and GHC (being right-angled at D and C) and having HAD = HGC (*by* 11. 3.) are equiangular; and ſo AD : GC :: AH : HG :: AE : AG (*by* 19. 4.); whence, alternately, AD : AE :: GC : AG, and $AD^2 : AE^2 :: GC^2$ (GF × HG) : AG^2 (GE × HG) :: GF : GE; therefore, *by diviſion*, &c. $AE^2 - AD^2 : AE^2 :: EF : GE :: EF \times AE : GE \times AE$; whence, again, *by alternation*, &c. $EF \times AE : AE^2 - AD^2$ ($\overline{AE+AD} \times \overline{AE-AD}$) :: GE × AE : AE^2 :: GE : AE :: radius : tang. AGE (*by Theor.* 2.); *from which the truth of the propoſition is manifeſt.*

Prop. XV.

If the relation of three right-lines a, b *and* x, *be ſuch, that* $ax - x^2 = b^2$; *then it will be, as* $\frac{1}{2}$ a : b :: *radius to the ſine of an angle; and, as radius, to the tangent* (*or co-tangent*) *of half this angle, ſo is* b ; x.

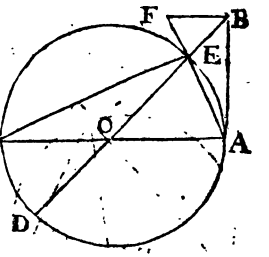


Make AB equal to *a*, upon which let a ſemicircle ADB be deſcribed; alſo let CD, equal to *b*, be perpendicular to AB, and meet the periphery in D (for it cannot exceed the radius of the circle when the propoſition is poſſible): moreover, let AD, BD, and the radius OD, be drawn. Becauſe AC × CB = $CD^2 = b^2$ (*by Cor. to* 19. 4.) it is plain that AC × $\overline{a-AC}$, or BC × $\overline{a-BC}$ is alſo = b^2; and, therefore, $x \times \overline{a-x}$ being = b^2, it is manifeſt that *x* may be equal, either, to AC, or to BC. Now (*by*

(*by Theor.* 1.) OD ($\frac{1}{2}a$) : CD (b) :: radius : fine DOC; whoſe half is equal to A, or BDC (*by* 10. 3.) But, as radius : tang. BDC :: DC (b) : BC; or, as radius : co-tang. BDC (tang. CDA) :: DC (b) : AC. *Q. E. D.*

PROP. XVI.

If the relation of three lines, a, b *and* x, *be ſuch, that* $x^2 \pm ax = b^2$; *then it will be, as* $\frac{1}{2}a$: b :: *radius to the tangent of an angle; and as radius is to the tangent, or co-tangent, of half this angle (according as the ſign of* ax *is poſitive or negative)* :: b : x.



Make AB = b, and AC, perpendicular to AB, equal to a; about the latter of which, as a diameter, let a circle be deſcribed; and, thro' O, the center thereof, let BD be drawn, meeting the periphery in E and D; alſo let A, E and C, E be joined, and draw BF parallel to AC, meeting AE, produced, in F. Then, ſince (*by* 22. 3.) BE × BD ($= BE \times \overline{BE + a} = BD \times \overline{BD - a}$) $= AB^2$ (b^2) and $x \times \overline{x \pm a} = b^2$, by ſuppoſition, it is manifeſt, that BE will be $= x$, when $x \times \overline{x + a} = b^2$; and BD $= x$, when $x \times \overline{x - a} = b^2$.

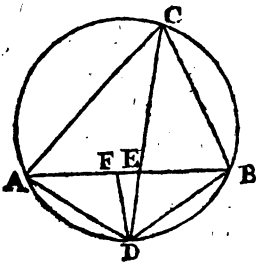
Furthermore, becauſe the angle F = OAE (*by* 7. 1.) = OEA (*by* 12. 1.) = BEF (*by* 3. 1.) it is evident that BF = BE (*by* 18. 1.) and that the angles BAF and C (being the complements of the equal angles F and OAE) are likewiſe equal.

Now (*by Theor.* 2.) AO ($\frac{1}{2}a$) : AB (b) :: radius : tang. AOB; whoſe half is equal to C, or BAF (*by* 14. 3.) But, as radius : tang. BAF :: AB (b) : BF

: BF (BE), the value of x in the first case, where $x^2 + ax = b^2$. Again, radius : co-tang. BAF (tang. F) : : BF (BE) : AB (*by Theor.* 2.); and BE : AB : : AB : BD (*by Cor. to* 22. 3. *and* 10. 4.); whence, by equality, radius : co-tang. BAF : : AB (*b*) : BD; which is the value of x in the second case; where $x^2 - ax = b^2$. *Q. E. D.*

Prop. XVII.

In any plane triangle ABC, *it will be, as the line* CE *bisecting the vertical angle, is to the base* AB, *so is the secant of half the vertical angle* ACB, *to the tangent of an angle; and, as the tangent of half this angle is to radius, so is the sine of half the vertical angle, to the sine of either angle, which the bisecting line makes with the base.*



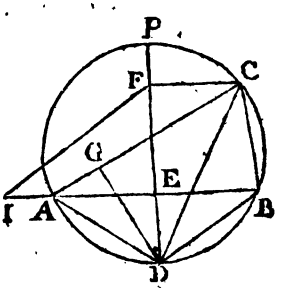
Let ACBD be a circle described about the triangle, and let CE be produced to meet the periphery thereof in D; moreover, let AD and BD be drawn, and likewise DF, perpendicular to the base AB; which will, also, bisect it, because (BCD being = ACD) the subtenses BD and AD are equal (*by* 10. 3.) Moreover, since the angle DBE = ACD (*by* 12. 3.) = DCB, the triangles DEB and DBC (having D common) are equiangular, and therefore $DE \times DC = DB^2$ (*by* 24. 3.) or, which is the same, $DE^2 + DE \times CE = DB^2$. Therefore (*by Prop.* 16.) CE : DB : : radius : tangent of an angle (which we will call Q); and, as radius : tang. $\frac{1}{2}$Q : : DB : DE.

But DB : BF ($\frac{1}{2}$AB) : : secant FBD (BCE) : radius; therefore, by compounding this, with the first

firſt proportion, we have, $\frac{1}{2}CE \times DB : \frac{1}{2}AB \times DB$: : radius × ſecant BCE : radius × tang. Q (*by* 10. 4.) and conſequently CE : AB : : ſecant BCE : tang. Q (*by* 7. 4.) Again, DE : DB : : ſine DBE (BCE) : ſine of DEB (or of CEB); whence, by equality, tang. $\frac{1}{2}$ Q : radius : : ſine BCE : ſine CEB. Q. *E. D.*

Prop. XVIII.

In any plane triangle ABC, *it will be, as the perpendicular is to the ſum of the two ſides, ſo is the tangent of half the angle at the vertex, to the tangent of an angle; and, as radius is to the tangent of half this angle, ſo is the ſum of the two ſides, to the baſe of the triangle.*



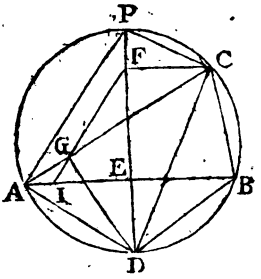
Let DP, perpendicular to AB, be the diameter of a circle deſcribed about the triangle; let CF be perpendicular to DP, and DG to AC, and let DA, DC and DB be drawn, and alſo FI, parallel to BD, meeting BA, produced, in I.

It is manifeſt, *from Prop.* 13. that CG is equal to half the ſum of the ſides AC and BC: it alſo appears, (*from* 7. 1. *and Cor. to* 12. 3.) that the triangles DCG, ADE, BDE, and IFE, are all equiangular. Therefore it will be, $CG^2 : BE^2 : : DC^2 (DF \times DP) : BD^2 (DE \times DP) : : DF (DE + EF) : DE : : BE + EI : EB : : \overline{BE + EI} \times BE : EB^2$; and, conſequently, $CG^2 (= \overline{BE + EI} \times BE) = BE^2 + EI \times BE$. But, *by Prop.* 16. it will be, $\frac{1}{2}EI : CG$: : radius : tangent of an angle (which we will call Q); and as radius : tang. $\frac{1}{2}$ Q : : CG : BE

(:: 2CG, or AC + BC, to AB). But (*by Theor.* 2.) EF : EI :: tang. I. (ACD) : radius; therefore, by compounding this proportion with the last but one, we shall have, $\frac{1}{2}$EI × EF : EI × CG :: tang. ACD × radius : tang. Q × radius (*by* 11. 4.) and consequently EF : 2CG (AC + BC) :: tang. ACD : tang. Q. *Whence the truth of the propoſition is manifeſt.*

Prop. XIX.

In any plane triangle ABC, *it will be, as the perpendicular is to the difference of the two ſides; ſo is the co-tangent of half the vertical angle, to the tangent of an angle; and, as radius is to the co-tangent of half this angle, ſo is the difference of the ſides to the baſe of the triangle.*



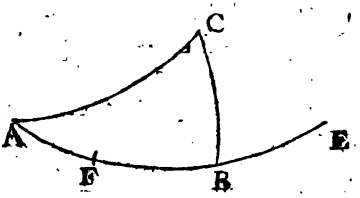
Let DP, DG, CF, &c. be as in the preceding propoſition; alſo let PA and PC be drawn, and FI, parallel to PA, meeting AB in I.

The right-angled triangles ADG and DPC, having DAG = DPC (*by Cor. to* 12. 3.) are ſimilar; and therefore, $AG^2 : PC^2 :: AD^2 : DP^2 :: AE^2 : AP^2$ (*by Cor. to* 11. 4.); whence, alternately, $AG^2 : AE^2 :: PC^2$ (PF × PD) : AP^2 (PE × PD) :: PF : PE :: AI : AE :: AI × AE : AE^2 (*by* 7. 4.); and conſequently, $AG^2 = AI \times AE = AE^2 - EI \times AE$. Therefore, *by Prop.* 16, $\frac{1}{2}$EI : AG :: radius : tangent of an angle ($\frac{1}{2}$Q); and as radius : co-tang. $\frac{1}{2}$Q :: AG : AE. But (*by Theor.* 2.) EF : EI :: co-tang. EFI (ACD) : radius; which proportion being compounded with the laſt but

but one, &c. we ſhall have, EF : 2 AG (AC—BC, *ſee Prop.* 13.) : : co-tang. EFI : tang. Q ; and as radius : co-tang. ½Q :: AG : AE : : 2AG (AC—BC) : 2AE (AB). *Q. E. D.*

PROP. XX.

The hypothenuſe AC, *and the ſum, or difference, of the legs* AB, BC, *of a right-angled ſpherical triangle* ABC, *being given, to determine the triangle.*



Let AE be the ſum, and AF the difference of the two legs. Becauſe, radius : co-ſ. AB : : co-ſ. BC : co-ſ. AC (*by Theor.* 2.) therefore, co-ſ. AB × co-ſ. BC = rad. × co-ſ. AC ; but the former of theſe is = ½ rad. × co-ſ. AE + co-ſ. AF (*by Corol.* 3. to *Prop.* 2.) ; therefore 2 × co-ſ. AC = co-ſ. AE + co-ſ. AF. Whence it appears, *that, if from twice the co-ſine of the hypothenuſe, the co-ſine of the given ſum, or difference, of the legs, be ſubtracted, the remainder will be the co-ſine of an arch, which added to the ſaid ſum, or difference, gives the double of the greater leg required.*

COROLLARY.

Hence, if the two legs be ſuppoſed equal to each other (or the given difference = 0), then will the co-ſine of the double of each, be equal to twice the co-ſine of the hypothenuſe *minus* the radius.

Prop. XXI.

One leg BC *and the ſum, or difference, of the hypothenuſe and the other leg* AB *being given, to determine the hypothenuſe (ſee the laſt figure.)*

Since rad. : co-ſine BC :: co-ſine AB : co-ſine AC (*by Theor.* 2.), it will be (*by comp. and div.*) radius + co-ſine BC : rad. — co-ſ. BC :: co-ſ. AB + co-ſ. AC : co-ſ. AB — co-ſ. AC. But the radius may be conſidered as the ſine of an arch of 90°, or the co-ſine of 0 : and, therefore, ſince (*by the lemma in p.* 30.) co-ſine o + co-ſine BC : co-ſ. o — co-ſ. BC :: co-tang. $\frac{BC + o}{2}$: tang. $\frac{BC - o}{2}$; and, co-ſine AB + co-ſ. AC : co-ſ. AB — co-ſ. AC :: co-tang. $\frac{AC + AB}{2}$: tang. $\frac{AC - AB}{2}$; it follows, *by equality*, that co-tang. $\frac{BC}{2}$: tang. $\frac{BC}{2}$ :: co-tang. $\frac{AC + AB}{2}$: tang. $\frac{AC - AB}{2}$; that is, *As the co-tang. of half the given leg, is to its tangent; ſo is the co-tang. of half the ſum of the hypothenuſe and the other leg, to the tangent of half their difference.*

Prop. XXII.

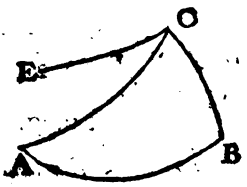
The angle at the baſe and the ſum, or difference, of the hypothenuſe and baſe, of a right-angled ſpherical triangle being given, to determine the triangle.

First,

First, it will be, rad. : co-ſ. A : : T. AC : T. AB (*by Theor.* 1.) and therefore rad. + co-ſ. A : rad. — co-ſ. A : : T. AC + T. AB : T. AC — T. AB : whence, by arguing as in the laſt Prop. it will appear, that, co-tang. ½A : tang. ½A : : rad. + co-ſ. A : rad. — co-ſ. A (: : T. AC + T. AB : T. AC — T. AB) : : S. $\overline{AC + AB}$: S. $\overline{AC - AB}$ (*by Prop.* 4.). Hence it appears, that, *As the co-tangent of half the given angle, is to its tangent; ſo is the ſine of the ſum of the hypothenuſe and adjacent leg, to the ſine of their difference.*

Prop. XXIII.

The hypothenuſe AC *and the ſum, or difference, of the two adjacent angles being given, to find the angles.*



Let EC be perpendicular to BC; and then it will be, rad. : co-ſ. AC : : T. A : T. ACE (*by Theor.* 5.) From whence, by reaſoning as above, we ſhall alſo have co-tang. ½AC : tang. ½AC : : S. $\overline{A + ACE}$: S. $\overline{A - ACE}$; whereof the two laſt terms, by ſubſtituting 90° — ACB for ACE, will become S. $\overline{90° + A - ACB}$ (co-ſ. ACB — A) and S. $\overline{A + ACB - 90°}$ reſpectively. Whence it appears, that, *As the co-tangent of half the hypothenuſe, is to its tangent; ſo is the co-ſine of the difference of the angles at the hypothenuſe, to the ſine of the exceſs of their ſum above a right-angle.*

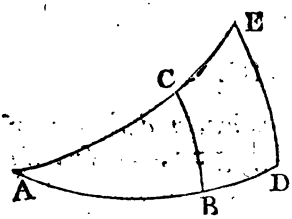
Corol-

COROLLARY.

Hence, if the angles be ſuppoſed equal, then it will be, as radius : tang. $\frac{1}{2}$AC : : tang. $\frac{1}{2}$AC : ſin. 2A — 90°.

PROP. XXIV.

In two right-angled ſpherical triangles ABC, ADE, *having one angle* A *common, let there be given the two perpendiculars* BC, DE *and the ſum, or difference, of the hypothenuſes* AC, AE, *to determine the triangles.*



It is evident (*from Theor.* 1.) that S. DE : S. BC : : S. AE. S. AC; therefore S. DE + S. BC : S. DE—S. BC : : S. AE + S. AC : S. AE — S. AC : whence (*by the lemma in p.* 30. *and equality*) tang. $\frac{DE + BC}{2}$: tang. $\frac{DE - BC}{2}$: : tang. $\frac{AE + AC}{2}$: tang. $\frac{AE - AC}{2}$: that is, *As the tangent of half the ſum of the two perpendiculars, is to the tangent of half their difference; ſo is the tangent of half the ſum of the two hypothenuſes, to the tangent of half their difference.*

PROP. XXV.

In two right-angled ſpherical triangles ABC, ADE, *having the ſame angle* A *at the baſe, let there be given the two perpendiculars* BC, DE *and the ſum or difference of the baſes* AB, AD, *to determine the baſes* (*ſee the preceding figure.*)

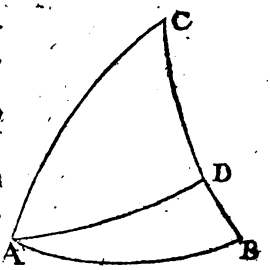
Since

Since T. DE : T. BC : : S. AD : S. AB (*by Theor. 4. and equality*); therefore is T. DE + T. BC : T. DE — T. BC : : S. AD + S. AB : S. AD — S. AB; whence, (*by Prop. 4. and the lemma in p. 30.*) it will be, S. $\overline{DE + BC}$: S. $\overline{DE - BC}$: : T. $\frac{AD + AB}{2}$: T. $\frac{AD - AB}{2}$; that is,

As the sine of the sum of the two perpendiculars, is to the sine of their difference; so is the tangent of half the sum of the two bases, to the tangent of half their difference.

PROP. XXVI.

The product of the square of radius and the co-sine of the base of any spherical triangle ABC, *is equal to the product of the sines of the two sides and the co-sine of the vertical angle, together with the product of radius and the co-sines of the same sides.*



For let AD be perpendicular to BC; then, since co-s. BD = $\frac{\text{S.CB} \times \text{S.CD} + \text{co-s.CB} \times \text{co-s.CD}}{\text{rad.}}$ (*by Cor. 1. to Prop. 2.*) it is evident, that co-s. BD : co-s. CD : : $\frac{\text{S.CB} \times \text{S.CD} + \text{co-s.CB} \times \text{co-s.CD}}{\text{rad.}}$: co-s. CD : : $\frac{\text{S. CB} \times \text{S. CD}}{\text{co-s. CD}}$ + co-s. CB : rad. (*by mult. each term by* $\frac{rad.}{co\text{-}s.CD}$). But $\frac{\text{S.CD}}{\text{co-s.CD}} = \frac{\text{T.CD}}{\text{rad.}}$, therefore our proportion will be $\frac{\text{S.CB} \times \text{T.CD}}{\text{rad.}}$ +

co-s.

co-f. CB : rad. (: : co-f. BD : co-f. CD) : : co-f. AB : co-f. AC (*by Corol. to Theor.* 2.) whence, by multiplying means and extremes, we have co-f. AB × radius $= \frac{\text{S. CB} \times \text{co-f. AC} \times \text{T. CD}}{\text{rad.}} +$ co-f. AC × co-f. BC. But (*by Theor.* 1.) radius : co-f. C : : T. AC : T. CD $= \frac{\text{co-f. C} \times \text{T. AC}}{\text{rad.}} =$ $\frac{\text{co-f. C} \times \text{S. AC}}{\text{co-f. AC}}$ (*by Corol.* 1. *Prop.* 1.) which laſt being ſubſtituted for its equal, we ſhall have,

co-f. AB × rad. $= \frac{\text{S. CA} \times \text{S. CB} \times \text{co-f. C}}{\text{rad.}} +$ co-f. AC × co-f. BC; from whence, if each term be multiplied by radius, the truth of the propoſition will appear manifeſt.

There is another way of demonſtrating this propoſition, from the orthographic projection of the ſphere; but that is a ſubject which neither room, nor inclination, will permit me to treat of here.

Prop. XXVII.

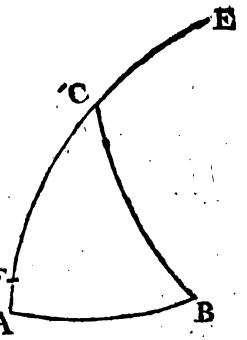
If AE *be the ſum, and* AF *the difference, of the two ſides of a ſpherical triangle* ABC, *and* V *be put to denote the verſed ſine of the vertical angle, and* R *the radius; then will*

$$\text{V} = \frac{\text{R}^2 \times \overline{\textit{co-f. AF} - \textit{co-f. AB}}}{\text{S. AC} \times \text{S. BC}}$$

$$= \frac{2\text{R} \times \overline{\textit{co-f. AF} - \textit{co-f. AB}}}{\overline{\textit{co-f. AF} - \textit{co-f. AE}}}$$

$$= \frac{2\text{R} \times \overline{\text{S. } \frac{1}{2}\text{AB} + \frac{1}{2}\text{AF}} \times \overline{\text{S. } \frac{1}{2}\text{AB} - \frac{1}{2}\text{AF}}}{\text{S. AC} \times \text{S. BC}}$$

It



It appears from the laſt Prop. that co-ſ. AB × R is $= \text{co-ſ. AC} \times \text{co-ſ. BC} + \frac{\text{S. AC} \times \text{S. BC} \times \text{co-ſ. C}}{\text{R}}$; in which, for co-ſ. C let its equal R — V be ſubſtituted, and then we ſhall have co-ſine AB × R = co-ſine AC × co-ſine BC + ſine AC × ſine BC — $\frac{\text{S. AC} \times \text{S. BC} \times \text{V}}{\text{R}}$: but the ſum of the two former of the three laſt terms is = co-ſ. AF × R (*by Cor.* 1. *to Prop.* 2.); therefore it will be co-ſ. AB × R = co-ſ. AF × R — $\frac{\text{S. AC} \times \text{S. BC} \times \text{V}}{\text{R}}$, and conſequently V = $\frac{\text{R}^2 \times \text{co-ſ. AF} - \text{co-ſ. AB}}{\text{S. AC} \times \text{S. BC}}$. *Which is the firſt caſe.* Again, becauſe S. AC × S. BC is = $\frac{1}{2}\text{R} \times \overline{\text{co-ſ. AF} - \text{co-ſ. AE}}$ (*by Corol.* 3. *to Prop.* 2.) we ſhall, alſo, have V = $\frac{2\text{R} \times \text{co-ſ. AF} - \text{co-ſ. AB}}{\text{co-ſ. AF} - \text{co-ſ. AE}}$ *Which is the ſecond caſe.* Moreover, ſince $\frac{1}{2}$ R × $\overline{\text{co-ſ. AF} - \text{co-ſ. AB}}$ is = S. $\frac{\text{AB} + \text{AF}}{2}$ × S. $\frac{\text{AB} - \text{AF}}{2}$ (*by the ſame*) it follows that V is likewiſe $= \frac{2\text{R} \times \text{S.} \overline{\frac{1}{2}\text{AB} + \frac{1}{2}\text{AF}} \times \text{S.} \overline{\frac{1}{2}\text{AB} - \frac{1}{2}\text{AF}}.}{\text{S. AC} \times \text{S. BC}}$

Q. E. D.

COROL.

Corollary 1.

Hence, because $\frac{1}{2}R \times V$ is = the square of the sine of $\frac{1}{2}C$ (*by Prop.* 1.) it follows that sq. S. $\frac{1}{2}C = \frac{R^2 \times \text{S. } \frac{1}{2}AB + \frac{1}{2}AF \times \text{S. } \frac{1}{2}AB - \frac{1}{2}AF}{\text{S. AC} \times \text{S. BC}}$

From whence we have the following theorem, for solving the 11th case of oblique triangles, where the three sides are given, to find an angle.

As the rectangle of the sines of the two sides, including the proposed angle, is to the rectangle under the sines of half the base plus *half the difference of the sides, and half the base* minus *half the difference of the sides; so is the square of radius, to the square of the sine of half the required angle.*

Corollary 2.

Moreover, because V is $= \frac{R^2 \times \text{co-s. AF} - \text{co-s. AB}}{\text{S. AC} \times \text{S. BC}}$; we shall have $R^2 : \text{S. AC} \times \text{S. BC} :: V : \text{co-s. AF} - \text{co-s. AB}$; which gives the following theorem, for finding a side, when the opposite angle, and the other two sides, are given.

As the square of radius, is to the rectangle of the sines of the two sides including the given angle; so is the versed sine of that angle, to the difference of the co-sines (or versed sines) of the difference of those sides, and the side required.

Corol-